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ORDINARY DIFFERENTIAL EQUATIONS

Ordinary Differential Equations

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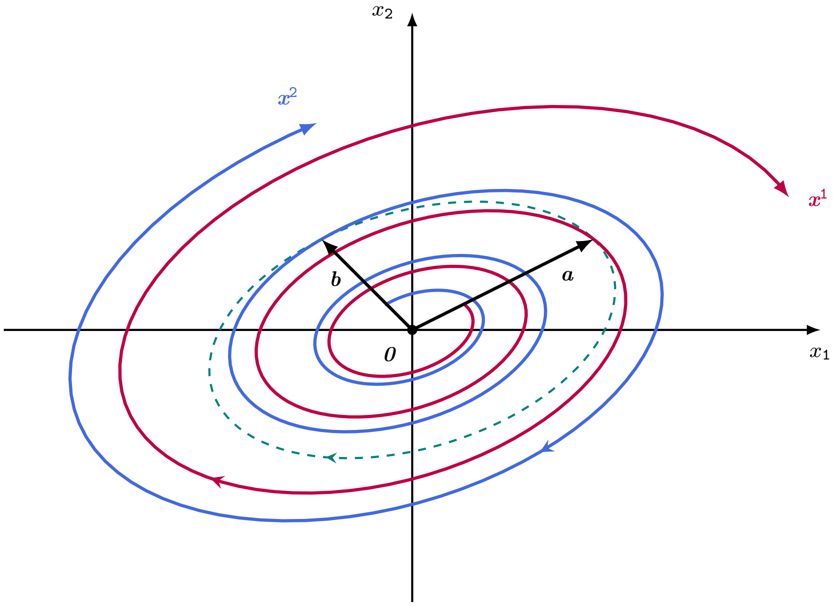


Contents

Preface 1

Chapter 1. First Order Equations 3
- 1.1. Linear Constant Coefficient Equations 4
- 1.1.1. Overview of Differential Equations 4
- 1.1.2. Linear Differential Equations 5
- 1.1.3. Solving Linear Differential Equations 6
- 1.1.4. The Integrating Factor Method 8
- 1.1.5. The Initial Value Problem 10
- 1.1.6. Exercises 13
- 1.2. Linear Variable Coefficient Equations 14
- 1.2.1. Review: Constant Coefficient Equations 14
- 1.2.2. Solving Variable Coefficient Equations 15
- 1.2.3. The Initial Value Problem 17
- 1.2.4. The Bernoulli Equation 19
- 1.2.5. Exercises 23
- 1.3. Separable Equations 24
- 1.3.1. Separable Equations 24
- 1.3.2. Euler Homogeneous Equations 29
- 1.3.3. Solving Euler Homogeneous Equations 32
- 1.3.4. Exercises 35
- 1.4. Exact Differential Equations 36
- 1.4.1. Exact Equations 36
- 1.4.2. Solving Exact Equations 37
- 1.4.3. Semi-Exact Equations 41
- 1.4.4. The Equation for the Inverse Function 46
- 1.4.5. Exercises 50
- 1.5. Applications of Linear Equations 51
- 1.5.1. Exponential Decay 51
- 1.5.2. Carbon-14 Dating 52
- 1.5.3. Newton's Cooling Law 53
- 1.5.4. Mixing Problems 54
- 1.5.5. Exercises 59
- 1.6. Nonlinear Equations 60
- 1.6.1. The Picard-Lindelöf Theorem 60
- 1.6.2. Comparison of Linear and Nonlinear Equations 69
- 1.6.3. Direction Fields 71
- 1.6.4. Exercises 75

Chapter 2. Second Order Linear Equations 77
- 2.1. Variable Coefficients 78

2.1.1. Definitions and Examples 78
2.1.2. Solutions to the Initial Value Problem. 80
2.1.3. Properties of Homogeneous Equations 81
2.1.4. The Wronskian Function 85
2.1.5. Abel's Theorem 86
2.1.6. Exercises 89
2.2. Reduction of Order Methods 90
2.2.1. Special Second Order Equations 90
2.2.2. Conservation of the Energy 93
2.2.3. The Reduction of Order Method 98
2.2.4. Exercises 101
2.3. Homogenous Constant Coefficients Equations 102
2.3.1. The Roots of the Characteristic Polynomial 102
2.3.2. Real Solutions for Complex Roots 106
2.3.3. Constructive Proof of Theorem 2.3.2 108
2.3.4. Exercises 111
2.4. Euler Equidimensional Equation 112
2.4.1. The Roots of the Indicial Polynomial 112
2.4.2. Real Solutions for Complex Roots 115
2.4.3. Transformation to Constant Coefficients 117
2.4.4. Exercises 119
2.5. Nonhomogeneous Equations 120
2.5.1. The General Solution Formula 120
2.5.2. The Undetermined Coefficients Method 121
2.5.3. The Variation of Parameters Method 125
2.5.4. Exercises 130
2.6. Applications 131
2.6.1. Review of Constant Coefficient Equations 131
2.6.2. Undamped Mechanical Oscillations 132
2.6.3. Damped Mechanical Oscillations 134
2.6.4. Electrical Oscillations 136
2.6.5. Exercises 139

Chapter 3. Power Series Solutions 141
3.1. Solutions Near Regular Points 143
3.1.1. Regular Points 143
3.1.2. The Power Series Method 144
3.1.3. The Legendre Equation 151
3.1.4. Exercises 155
3.2. Solutions Near Regular Singular Points 156
3.2.1. Regular Singular Points 156
3.2.2. The Frobenius Method 159
3.2.3. The Bessel Equation 163
3.2.4. Exercises 168
Notes on Chapter 3 169

Chapter 4. The Laplace Transform Method 173
4.1. Introduction to the Laplace Transform 175
4.1.1. Oveview of the Method 175
4.1.2. The Laplace Transform 176

4.1.3. Main Properties 180
4.1.4. Solving Differential Equations 184
4.1.5. Exercises 186
4.2. The Initial Value Problem 187
4.2.1. Solving Differential Equations 187
4.2.2. One-to-One Property 188
4.2.3. Partial Fractions 190
4.2.4. Higher Order IVP 195
4.2.5. Exercises 197
4.3. Discontinuous Sources 198
4.3.1. Step Functions 198
4.3.2. The Laplace Transform of Steps 199
4.3.3. Translation Identities 200
4.3.4. Solving Differential Equations 204
4.3.5. Exercises 209
4.4. Generalized Sources 210
4.4.1. Sequence of Functions and the Dirac Delta 210
4.4.2. Computations with the Dirac Delta 212
4.4.3. Applications of the Dirac Delta 214
4.4.4. The Impulse Response Function 215
4.4.5. Comments on Generalized Sources 218
4.4.6. Exercises 221
4.5. Convolutions and Solutions 222
4.5.1. Definition and Properties 222
4.5.2. The Laplace Transform 224
4.5.3. Solution Decomposition 226
4.5.4. Exercises 230

Chapter 5. Systems of Linear Differential Equations 231
5.1. General Properties 232
5.1.1. First Order Linear Systems 232
5.1.2. Existence of Solutions 234
5.1.3. Order Transformations 235
5.1.4. Homogeneous Systems 238
5.1.5. The Wronskian and Abel’s Theorem 242
5.1.6. Exercises 246
5.2. Solution Formulas 247
5.2.1. Homogeneous Systems 247
5.2.2. Homogeneous Diagonalizable Systems 249
5.2.3. Nonhomogeneous Systems 256
5.2.4. Exercises 259
5.3. Two-Dimensional Homogeneous Systems 260
5.3.1. Diagonalizable Systems 260
5.3.2. Non-Diagonalizable Systems 263
5.3.3. Exercises 266
5.4. Two-Dimensional Phase Portraits 267
5.4.1. Real Distinct Eigenvalues 268
5.4.2. Complex Eigenvalues 271
5.4.3. Repeated Eigenvalues 273
5.4.4. Exercises 275

Chapter 6. Autonomous Systems and Stability 277
6.1. Flows on the Line 279
6.1.1. Autonomous Equations 279
6.1.2. Geometrical Characterization of Stability 281
6.1.3. Critical Points and Linearization 283
6.1.4. Population Growth Models 286
6.1.5. Exercises 290
6.2. Flows on the Plane 291
6.2.1. Two-Dimensional Nonlinear Systems 291
6.2.2. Review: The Stability of Linear Systems 292
6.2.3. Critical Points and Linearization 294
6.2.4. The Stability of Nonlinear Systems 297
6.2.5. Competing Species 299
6.2.6. Exercises 302

Chapter 7. Boundary Value Problems 303
7.1. Eigenfunction Problems 304
7.1.1. Two-Point Boundary Value Problems 304
7.1.2. Comparison: IVP and BVP 305
7.1.3. Eigenfunction Problems 308
7.1.4. Exercises 312
7.2. Overview of Fourier series 313
7.2.1. Fourier Expansion of Vectors 313
7.2.2. Fourier Expansion of Functions 315
7.2.3. Even or Odd Functions 320
7.2.4. Sine and Cosine Series 321
7.2.5. Applications 324
7.2.6. Exercises 326
7.3. The Heat Equation 327
7.3.1. The Heat Equation (in One-Space Dim) 327
7.3.2. The IBVP: Dirichlet Conditions 329
7.3.3. The IBVP: Neumann Conditions 332
7.3.4. Exercises 339

Chapter 8. Review of Linear Algebra 341
8.1. Linear Algebraic Systems 342
8.1.1. Systems of Linear Equations 342
8.1.2. Gauss Elimination Operations 346
8.1.3. Linearly Dependence 349
8.1.4. Exercises 350
8.2. Matrix Algebra 351
8.2.1. A Matrix is a Function 351
8.2.2. Matrix Operations 352
8.2.3. The Inverse Matrix 356
8.2.4. Computing the Inverse Matrix 358
8.2.5. Overview of Determinants 359
8.2.6. Exercises 362
8.3. Eigenvalues and Eigenvectors 363
8.3.1. Eigenvalues and Eigenvectors 363
8.3.2. Diagonalizable Matrices 370

8.3.3. Exercises 375
8.4. The Matrix Exponential 376
8.4.1. The Exponential Function 376
8.4.2. Diagonalizable Matrices Formula 378
8.4.3. Properties of the Exponential 379
8.4.4. Exercises 383

Chapter 9. Appendices 385
A. Overview of Complex Numbers 385
A.1. Extending the Real Numbers 387
A.2. The Imaginary Unit 387
A.3. Standard Notation 388
A.4. Useful Formulas 389
A.5. Complex Functions 391
A.6. Complex Vectors 393
B. Overview of Power Series 396
C. Discrete and Continuum Equations 400
C.1. The Difference Equation 400
C.2. Solving the Difference Equation 402
C.3. The Differential Equation 403
C.4. Solving the Differential Equation 404
C.5. Summary and Consistency 405
C.6. Exercises 409
D. Review Exercises 411
E. Practice Exams 411
F. Answers to exercises 412

Bibliography 423

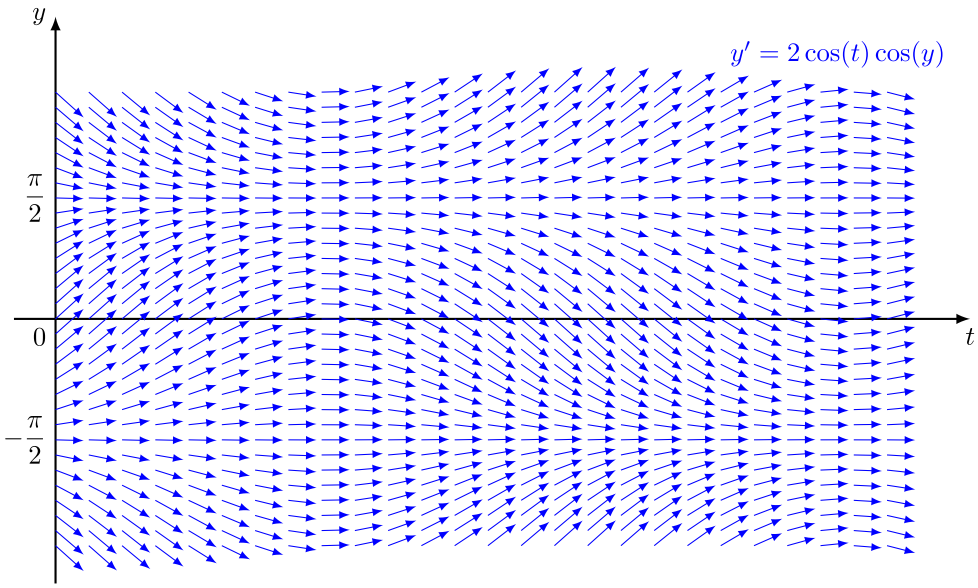
Preface

This is an introduction to ordinary differential equations. We describe the main ideas to solve certain differential equations, such us first order scalar equations, second order linear equations, and systems of linear equations. We use power series methods to solve variable coefficients second order linear equations. We introduce Laplace transform methods to find solutions to constant coefficients equations with generalized source functions. We provide a brief introduction to boundary value problems, eigenvalue-eigenfunction problems, and Fourier series expansions. We end these notes solving our first partial differential equation, the heat equation. We use the method of separation of variables, where solutions to the partial differential equation are obtained by solving infinitely many ordinary differential equations.

CHAPTER 1

First Order Equations

We start our study of differential equations in the same way the pioneers in this field did. We show particular techniques to solve particular types of first order differential equations. The techniques were developed in the eighteenth and nineteenth centuries and the equations include linear equations, separable equations, Euler homogeneous equations, and exact equations. This way of studying differential equations reached a dead end pretty soon. Most of the differential equations cannot be solved by any of the techniques presented in the first sections of this chapter. People then tried something different. Instead of solving the equations they tried to show whether an equation has solutions or not, and what properties such solution may have. This is less information than obtaining the solution, but it is still valuable information. The results of these efforts are shown in the last sections of this chapter. We present theorems describing the existence and uniqueness of solutions to a wide class of first order differential equations.



1.1. Linear Constant Coefficient Equations

1.1.1. Overview of Differential Equations. A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. Differential equations are essential for a mathematical description of nature—they lie at the core of many physical theories. For example, let us just mention Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schrödinger's equation for quantum mechanics, and Einstein's equation for the general theory of gravitation. We now show what differential equations look like.

Example 1.1.1.

(a) **Newton's law:** Mass times acceleration equals force, $ma = f$, where m is the particle mass, $a = d^2x/dt^2$ is the particle acceleration, and f is the force acting on the particle. Hence Newton's law is the differential equation

$$m\frac{d^2\boldsymbol{x}}{dt^2}(t) = \boldsymbol{f}\Big(t, \boldsymbol{x}(t), \frac{d\boldsymbol{x}}{dt}(t)\Big),$$

where the unknown is $\boldsymbol{x}(t)$—the position of the particle in space at the time t. As we see above, the force may depend on time, on the particle position in space, and on the particle velocity.

Remark: This is a second order Ordinary Differential Equation (ODE).

(b) **Radioactive Decay:** The amount u of a radioactive material changes in time as follows,

$$\frac{du}{dt}(t) = -k\,u(t), \qquad k > 0,$$

where k is a positive constant representing radioactive properties of the material.

Remark: This is a first order ODE.

(c) **The Heat Equation:** The temperature T in a solid material changes in time and in three space dimensions—labeled by $\boldsymbol{x} = (x, y, z)$—according to the equation

$$\frac{\partial T}{\partial t}(t,\boldsymbol{x}) = k\Big(\frac{\partial^2 T}{\partial x^2}(t,\boldsymbol{x}) + \frac{\partial^2 T}{\partial y^2}(t,\boldsymbol{x}) + \frac{\partial^2 T}{\partial z^2}(t,\boldsymbol{x})\Big), \qquad k > 0,$$

where k is a positive constant representing thermal properties of the material.

Remark: This is a first order in time and second order in space PDE.

(d) **The Wave Equation:** A wave perturbation u propagating in time t and in three space dimensions—labeled by $\boldsymbol{x} = (x, y, z)$—through the media with wave speed $v > 0$ is

$$\frac{\partial^2 u}{\partial t^2}(t,\boldsymbol{x}) = v^2\Big(\frac{\partial^2 u}{\partial x^2}(t,\boldsymbol{x}) + \frac{\partial^2 u}{\partial y^2}(t,\boldsymbol{x}) + \frac{\partial^2 u}{\partial z^2}(t,\boldsymbol{x})\Big).$$

Remark: This is a second order in time and space Partial Differential Equation (PDE). ◁

The equations in examples (a) and (b) are called *ordinary differential equations* (ODE)— the unknown function depends on a single independent variable, t. The equations in examples (d) and (c) are called *partial differential equations* (PDE)—the unknown function depends on two or more independent variables, t, x, y, and z, and their partial derivatives appear in the equations.

The *order* of a differential equation is the highest derivative order that appears in the equation. Newton's equation in example (a) is second order, the time decay equation in example (b) is first order, the wave equation in example (d) is second order is time and space variables, and the heat equation in example (c) is first order in time and second order in space variables.

1.1.2. Linear Differential Equations. We start with a precise definition of a first order ordinary differential equation. Then we introduce a particular type of first order equations—linear equations.

Definition 1.1.1. *A* ***first order ODE*** *on the unknown y is*

$$y'(t) = f(t, y(t)), \tag{1.1.1}$$

where f is given and $y' = \dfrac{dy}{dt}$. The equation is ***linear*** *iff the source function f is linear on its second argument,*

$$y' = a(t)\,y + b(t). \tag{1.1.2}$$

The linear equation has ***constant coefficients*** *iff both a and b above are constants. Otherwise the equation has* ***variable coefficients****.*

There are different sign conventions for Eq. (1.1.2) in the literature. For example, Boyce-DiPrima [3] writes it as $y' = -a\,y + b$. The sign choice in front of function a is matter of taste. Some people like the negative sign, because later on, when they write the equation as $y' + a\,y = b$, they get a plus sign on the left-hand side. In any case, we stick here to the convention $y' = ay + b$.

Example 1.1.2.

(a) An example of a first order linear ODE is the equation

$$y' = 2\,y + 3.$$

On the right-hand side we have the function $f(t, y) = 2y + 3$, where we can see that $a(t) = 2$ and $b(t) = 3$. Since these coefficients do not depend on t, this is a constant coefficient equation.

(b) Another example of a first order linear ODE is the equation

$$y' = -\frac{2}{t}\,y + 4t.$$

In this case, the right-hand side is given by the function $f(t, y) = -2y/t + 4t$, where $a(t) = -2/t$ and $b(t) = 4t$. Since the coefficients are nonconstant functions of t, this is a variable coefficients equation.

(c) The equation $y' = -\dfrac{2}{ty} + 4t$ is nonlinear.

◁

We denote by $y : D \subset \mathbb{R} \to \mathbb{R}$ a real-valued function y defined on a domain D. Such a function is *solution* of the differential equation (1.1.1) iff the equation is satisfied for all values of the independent variable t on the domain D.

Example 1.1.3. Show that $y(t) = e^{2t} - \dfrac{3}{2}$ is solution of the equation

$$y' = 2\,y + 3.$$

Solution: We need to compute the left and right-hand sides of the equation and verify they agree. On the one hand we compute $y'(t) = 2e^{2t}$. On the other hand we compute

$$2\,y(t) + 3 = 2\Big(e^{2t} - \frac{3}{2}\Big) + 3 = 2e^{2t}.$$

We conclude that $y'(t) = 2\,y(t) + 3$ for all $t \in \mathbb{R}$. ◁

Example 1.1.4. Find the differential equation $y' = f(y)$ satisfied by $y(t) = 4\,e^{2t} + 3$.

Solution: **(Solution Video)** We compute the derivative of y,

$$y' = 8\,e^{2t}$$

We now write the right-hand side above, in terms of the original function y, that is,

$$y = 4\,e^{2t} + 3 \quad \Rightarrow \quad y - 3 = 4\,e^{2t} \quad \Rightarrow \quad 2(y-3) = 8\,e^{2t}.$$

So we got a differential equation satisfied by y, namely

$$y' = 2y - 6.$$

◁

1.1.3. Solving Linear Differential Equations. Linear equations with constant coefficient are simpler to solve than variable coefficient ones. But integrating each side of the equation *does not* work. For example, take the equation

$$y' = 2\,y + 3,$$

and integrate with respect to t on both sides,

$$\int y'(t)\,dt = 2\int y(t)\,dt + 3t + c, \qquad c \in \mathbb{R}.$$

The Fundamental Theorem of Calculus implies $y(t) = \int y'(t)\,dt$, so we get

$$y(t) = 2\int y(t)\,dt + 3t + c.$$

Integrating both sides of the differential equation is not enough to find a solution y. We still need to find a primitive of y. We have only rewritten the original differential equation as an integral equation. Simply integrating both sides of a linear equation does not solve the equation.

We now state a precise formula for the solutions of constant coefficient linear equations. The proof relies on a new idea—a clever use of the chain rule for derivatives.

Theorem 1.1.2 (Constant Coefficients)**.** *The linear differential equation*

$$y' = a\,y + b \tag{1.1.3}$$

with $a \neq 0$, b constants, has infinitely many solutions,

$$y(t) = c\,e^{at} - \frac{b}{a}, \qquad c \in \mathbb{R}. \tag{1.1.4}$$

Remarks:

(a) Equation (1.1.4) is called the *general solution* of the differential equation in (1.1.3).

(b) Theorem 1.1.2 says that Eq. (1.1.3) has infinitely many solutions, one solution for each value of the constant c, which is not determined by the equation.

(c) It makes sense that we have a free constant c in the solution of the differential equation. The differential equation contains a first derivative of the unknown function y, so finding a solution of the differential equation requires one integration. Every indefinite integration introduces an integration constant. This is the origin of the constant c above.

Proof of Theorem 1.1.2: First consider the case $b = 0$, so $y' = a\,y$, with $a \in \mathbb{R}$. Then,

$$y' = a\,y \quad \Rightarrow \quad \frac{y'}{y} = a \quad \Rightarrow \quad \ln(|y|)' = a \quad \Rightarrow \quad \ln(|y|) = at + c_0,$$

where $c_0 \in \mathbb{R}$ is an arbitrary integration constant, and we used the Fundamental Theorem of Calculus on the last step, $\int \ln(|y|)'\,dt = \ln(|y|)$. Compute the exponential on both sides,

$$y(t) = \pm e^{at+c_0} = \pm e^{c_0}\,e^{at}, \quad \text{denote} \quad c = \pm e^{c_0} \quad \Rightarrow \quad y(t) = c\,e^{at}, \quad c \in \mathbb{R}.$$

This is the solution of the differential equation in the case that $b = 0$. The case $b \neq 0$ can be converted into the case above. Indeed,

$$y' = a\,y + b \quad \Rightarrow \quad y' = a\left(y + \frac{b}{a}\right) \quad \Rightarrow \quad \left(y + \frac{b}{a}\right)' = a\left(y + \frac{b}{a}\right),$$

since $(b/a)' = 0$. Denoting $\tilde{y} = y + (b/a)$, the equation above is $\tilde{y}' = a\,\tilde{y}$. We know all the solutions to that equation,

$$\tilde{y}(t) = c\,e^{at}, \quad c \in \mathbb{R} \quad \Rightarrow \quad y(t) + \frac{b}{a} = c\,e^{at} \quad \Rightarrow \quad y(t) = c\,e^{at} - \frac{b}{a}.$$

This establishes the Theorem. □

Remark: We solved the differential equation above, $y' = a\,y$, by transforming it into a total derivative. Let us highlight this fact in the calculation we did,

$$\ln(|y|)' = a \quad \Rightarrow \quad \bigl(\ln(|y|) - at\bigr)' = 0 \quad \Leftrightarrow \quad \psi(t, y(t))' = 0, \quad \text{with} \quad \psi = \ln(|y(t)|) - at.$$

The function ψ is called a *potential function*. This is how the original differential equation gets transformed into a total derivative,

$$y' = a\,y \quad \to \quad \psi' = 0.$$

Total derivatives are simple to integrate,

$$\psi' = 0 \quad \Rightarrow \quad \psi = c_0, \qquad c_0 \in \mathbb{R}.$$

So the solution is

$$\ln(|y|) - at = c_0 \quad \Rightarrow \quad \ln(|y|) = c_0 + at \quad \Rightarrow \quad y(t) = \pm e^{c_0+at} = \pm e^{c_0}\,e^{at},$$

and denoting $c = \pm e^{c_0}$ we reobtain the formula

$$y(t) = c\,e^{at}.$$

In the case $b \neq 0$ a potential function is $\psi(t, y(t)) = \ln\Bigl(\bigl|y(t) + \frac{b}{a}\bigr|\Bigr) - at$.

Example 1.1.5. Find all solutions to the constant coefficient equation $y' = 2y + 3$.

Solution: **(Solution Video)** Let's pull a common factor 2 on the right-hand side of the equation,

$$y' = 2\Bigl(y + \frac{3}{2}\Bigr) \quad \Rightarrow \quad \Bigl(y + \frac{3}{2}\Bigr)' = 2\,\Bigl(y + \frac{3}{2}\Bigr).$$

Denoting $\tilde{y} = y + (3/2)$ we get

$$\tilde{y}' = 2\,\tilde{y} \quad \Rightarrow \quad \frac{\tilde{y}'}{\tilde{y}} = 2 \quad \Rightarrow \quad \ln(|\tilde{y}|)' = 2 \quad \Rightarrow \quad \ln(|\tilde{y}|) = 2t + c_0.$$

We now compute exponentials on both sides, to get

$$\tilde{y}(t) = \pm e^{2t+c_0} = \pm e^{2t}\, e^{c_0}, \quad \text{denote} \quad c = \pm e^{c_0}, \quad \text{then} \quad \tilde{y}(t) = c\, e^{2t}, \quad c \in \mathbb{R}.$$

Since $\tilde{y} = y + \dfrac{3}{2}$, we get $y(t) = c\, e^{2t} - \dfrac{3}{2}$, where $c \in \mathbb{R}$. ◁

Remark: We converted the original differential equation $y' = 2\,y + 3$ into a total derivative of a potential function $\psi' = 0$. The potential function can be computed from the step

$$\ln(|\tilde{y}|)' = 2 \quad \Rightarrow \quad \big(\ln(|\tilde{y}|) - 2t\big)' = 0,$$

then a potential function is $\psi(t, y(t)) = \ln\Big(\big|y(t) + \dfrac{3}{2}\big|\Big) - 2t$. Since the equation is now $\psi' = 0$, all solutions are $\psi = c_0$, with $c_0 \in \mathbb{R}$. That is

$$\ln\Big(\big|y(t) + \frac{3}{2}\big|\Big) - 2t = c_0 \quad \Rightarrow \quad \ln\Big(\big|y(t) + \frac{3}{2}\big|\Big) = 2t + c_0 \quad \Rightarrow \quad y(t) + \frac{3}{2} = \pm e^{2t+c_0}.$$

If we denote $c = \pm e^{c_0}$, then we get the solution we found above, $y(t) = c\, e^{2t} - \dfrac{3}{2}$.

1.1.4. The Integrating Factor Method.

The argument we used to prove Theorem 1.1.2 cannot be generalized in a simple way to all linear equations with *variable* coefficients. However, there is a way to solve linear equations with both constant and variable coefficients—the *integrating factor method*. Now we give a second proof of Theorem 1.1.2 using this method.

Second Proof of Theorem 1.1.2: Write the equation with y on one side only,

$$y' - a\, y = b,$$

and then multiply the differential equation by a function μ, called an integrating factor,

$$\mu\, y' - a\, \mu\, y = \mu\, b. \tag{1.1.5}$$

Now comes the critical step. We choose a *positive* function μ such that

$$-\, a\, \mu = \mu'. \tag{1.1.6}$$

For any function μ solution of Eq. (1.1.6), the differential equation in (1.1.5) has the form

$$\mu\, y' + \mu'\, y = \mu\, b.$$

But the left-hand side is a total derivative of a product of two functions,

$$\big(\mu\, y\big)' = \mu\, b. \tag{1.1.7}$$

This is the property we want in an integrating factor, μ. We want to find a function μ such that the left-hand side of the differential equation for y can be written as a total derivative, just as in Eq. (1.1.7). We only need to find one of such functions μ. So we go back to Eq. (1.1.6), the differential equation for μ, which is simple to solve,

$$\mu' = -a\, \mu \quad \Rightarrow \quad \frac{\mu'}{\mu} = -a \quad \Rightarrow \quad \big(\ln(|\mu|)\big)' = -a \quad \Rightarrow \quad \ln(|\mu|) = -at + c_0.$$

Computing the exponential of both sides in the equation above we get

$$\mu = \pm e^{c_0 - at} = \pm e^{c_0}\, e^{-at} \quad \Rightarrow \quad \mu = c_1\, e^{-at}, \qquad c_1 = \pm e^{c_0}.$$

Since c_1 is a constant which will cancel out from Eq. (1.1.5) anyway, we choose the integration constant $c_0 = 0$, hence $c_1 = 1$. The integrating function is then

$$\mu(t) = e^{-at}.$$

This function is an integrating factor, because if we start again at Eq. (1.1.5), we get

$$e^{-at}\,y' - a\,e^{-at}\,y = b\,e^{-at} \quad \Rightarrow \quad e^{-at}\,y' + \big(e^{-at}\big)'\,y = b\,e^{-at},$$

where we used the main property of the integrating factor, $-a\,e^{-at} = \big(e^{-at}\big)'$. Now the product rule for derivatives implies that the left-hand side above is a total derivative,

$$\big(e^{-at}\,y\big)' = b\,e^{-at}.$$

The right-hand side above can be rewritten as a derivative, $b\,e^{-at} = \Big(-\dfrac{b}{a}\,e^{-at}\Big)'$, hence

$$\Big(e^{-at}\,y + \frac{b}{a}\,e^{-at}\Big)' = 0 \quad \Leftrightarrow \quad \Big[\Big(y + \frac{b}{a}\Big)\,e^{-at}\Big]' = 0.$$

We have succeeded in writing the *whole differential equation as a total derivative.* The differential equation is the total derivative of a *potential function*, which in this case is

$$\psi(t,y) = \Big(y + \frac{b}{a}\Big)\,e^{-at}.$$

Notice that this potential function is the exponential of the potential function found in the first proof of this Theorem. The differential equation for y is a total derivative,

$$\frac{d\psi}{dt}(t, y(t)) = 0,$$

so it is simple to integrate,

$$\psi(t, y(t)) = c \quad \Rightarrow \quad \Big(y(t) + \frac{b}{a}\Big)\,e^{-at} = c \quad \Rightarrow \quad y(t) = c\,e^{at} - \frac{b}{a}.$$

This establishes the Theorem. ☐

We solve the example below following the second proof of Theorem 1.1.2.

Example 1.1.6. Find all solutions to the constant coefficient equation

$$y' = 2y + 3 \tag{1.1.8}$$

Solution: **(Solution Video)** Write the equation in (1.1.8) as follows,

$$y' - 2y = 3.$$

Multiply this equation by the integrating factor $\mu(t) = e^{-2t}$,

$$e^{-2t}y' - 2\,e^{-2t}\,y = 3\,e^{-2t} \quad \Leftrightarrow \quad e^{-2t}y' + \big(e^{-2t}\big)'\,y = 3\,e^{-2t}.$$

We now solve the same problem above, but now using the formulas in Theorem 1.1.2.

Example 1.1.7. Find all solutions to the constant coefficient equation

$$y' = 2y + 3 \tag{1.1.9}$$

Solution: The equation above is the case of $a = 2$ and $b = 3$ in Eq. (1.1.3). Therefore, using these values in the expression for the solution given in Eq. (1.1.4) we obtain

$$y(t) = c\,e^{2t} - \frac{3}{2}.$$

◁

The equation on the far right above is

$$(e^{-2t}\,y)' = 3\,e^{-2t}.$$

Rewrite the right-hand side above,

$$(e^{-2t}\,y)' = \Big(-\frac{3}{2}\,e^{-2t}\Big)'.$$

Moving terms and reordering factors we get

$$\Big[\Big(y + \frac{3}{2}\Big)\,e^{-2t}\Big]' = 0.$$

The equation is a total derivative, $\psi' = 0$, of the potential function

$$\psi(t,y) = \Big(y + \frac{3}{2}\Big)\,e^{-2t}.$$

Now the equation is easy to integrate,

$$\Big(y + \frac{3}{2}\Big)\,e^{-2t} = c.$$

So we get the solutions

$$y(t) = c\,e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}. \qquad \lhd$$

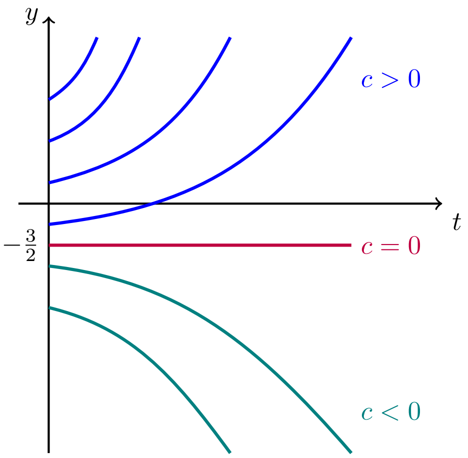


FIGURE 1. A few solutions to Eq. (1.1.8) for different c. **(Interactive Graph)**

1.1.5. The Initial Value Problem. Sometimes in physics one is not interested in all solutions to a differential equation, but only in those solutions satisfying extra conditions. For example, in the case of Newton's second law of motion for a point particle, one could be interested only in solutions such that the particle is at a specific position at the initial time. Such condition is called an initial condition, and it selects a subset of solutions of the differential equation. An initial value problem means to find a solution to both a differential equation and an initial condition.

Definition 1.1.3. *The **initial value problem** (IVP) is to find all solutions y to*

$$y' = a\,y + b, \tag{1.1.10}$$

that satisfy the initial condition

$$y(t_0) = y_0, \tag{1.1.11}$$

where a, b, t_0, and y_0 are given constants.

Remark: The equation (1.1.11) is called the *initial condition* of the problem.

Although the differential equation in (1.1.10) has infinitely many solutions, the associated initial value problem has a unique solution.

Theorem 1.1.4 (Constant Coefficients IVP)**.** *Given the constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem*

$$y' = a\,y + b, \qquad y(t_0) = y_0,$$

has the unique solution

$$y(t) = \Big(y_0 + \frac{b}{a}\Big)\,e^{a(t-t_0)} - \frac{b}{a}. \tag{1.1.12}$$

Remark: In case $t_0 = 0$ the initial condition is $y(0) = y_0$ and the solution is

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{at} - \frac{b}{a}.$$

The proof of Theorem 1.1.4 is just to write the general solution of the differential equation given in Theorem 1.1.2, and fix the integration constant c with the initial condition.

Proof of Theorem 1.1.4: The general solution of the differential equation in (1.1.10) is given in Eq. (1.1.4) for any choice of the integration constant c,

$$y(t) = c\, e^{at} - \frac{b}{a}.$$

The initial condition determines the value of the constant c, as follows

$$y_0 = y(t_0) = c\, e^{at_0} - \frac{b}{a} \quad \Leftrightarrow \quad c = \left(y_0 + \frac{b}{a}\right) e^{-at_0}.$$

Introduce this expression for the constant c into the differential equation in Eq. (1.1.10),

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}.$$

This establishes the Theorem. □

Example 1.1.8. Find the unique solution of the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1. \tag{1.1.13}$$

Solution: (Solution Video) All solutions of the differential equation are given by

$$y(t) = ce^{2t} - \frac{3}{2},$$

where c is an arbitrary constant. The initial condition in Eq. (1.1.13) determines c,

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}.$$

Then, the unique solution to the initial value problem above is $y(t) = \frac{5}{2}\, e^{2t} - \frac{3}{2}$. ◁

Example 1.1.9. Find the solution y to the initial value problem

$$y' = -3y + 1, \qquad y(0) = 1.$$

Solution: (Solution Video) Write the differential equation as $y' + 3\,y = 1$, and multiply the equation by the integrating factor $\mu = e^{3t}$, which will convert the left-hand side above into a total derivative,

$$e^{3t}y' + 3\,e^{3t}\,y = e^{3t} \quad \Leftrightarrow \quad e^{3t}y' + \left(e^{3t}\right)'\,y = e^{3t}.$$

This is the key idea, because the derivative of a product implies

$$\left(e^{3t}\,y\right)' = e^{3t}.$$

The exponential e^{3t} is an integrating factor. Integrate on both sides of the equation,

$$e^{3t}\,y = \frac{1}{3}e^{3t} + c.$$

So every solution of the differential equation above is given by

$$y(t) = c\,e^{-3t} + \frac{1}{3}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 2$ selects only one solution,

$$1 = y(0) = c + \frac{1}{3} \quad \Rightarrow \quad c = \frac{2}{3}.$$

We get the solution $y(t) = \frac{2}{3}\, e^{-3t} + \frac{1}{3}$. ◁

Notes. This section corresponds to Boyce-DiPrima [3] Section 2.1, where both constant and variable coefficient equations are studied. Zill and Wright give a more concise exposition in [17] Section 2.3, and a one page description is given by Simmons in [10] in Section 2.10. The integrating factor method is shown in most of these books, but unlike them, here we emphasize that the integrating factor changes the linear differential equation into a total derivative, which is trivial to integrate. We also show here how to compute the potential functions for the linear differential equations. In § 1.4 we solve (nonlinear) exact equations and nonexact equations with integrating factors. We solve these equations by transforming them into a total derivative, just as we did in this section with the linear equations.

1.1.6. Exercises.

1.1.1.- Find the differential equation of the form $y' = f(y)$ satisfied by the function

$$y(t) = 8e^{5t} - \frac{2}{5}.$$

1.1.2.- Find constants a, b, so that

$$y(t) = (t+3)\, e^{2t}$$

is solution of the IVP

$$y' = ay + e^{2t}, \quad y(0) = b.$$

1.1.3.- Find all solutions y of

$$y' = 3y.$$

1.1.4.- Follow the steps below to find all solutions of

$$y' = -4y + 2$$

(a) Find the integrating factor μ.

(b) Write the equations as a total derivative of a function ψ, that is

$$y' = -4y + 2 \quad \Leftrightarrow \quad \psi' = 0.$$

(c) Integrate the equation for ψ.

(d) Compute y using part (c).

1.1.5.- Find all solutions of

$$y' = 2y + 5$$

1.1.6.- Find the solution of the IVP

$$y' = -4y + 2, \quad y(0) = 5.$$

1.1.7.- Find the solution of the IVP

$$\frac{dy}{dt}(t) = 3\, y(t) - 2, \quad y(1) = 1.$$

1.1.8.- Express the differential equation

$$y' = 6\, y + 1 \tag{1.1.14}$$

as a total derivative of a potential function $\psi(t, y)$, that is, find ψ satisfying

$$y' = 6\, y + 1 \quad \Leftrightarrow \quad \psi' = 0.$$

Integrate the equation for the potential function ψ to find all solutions y of Eq. (1.1.14).

1.1.9.- Find the solution of the IVP

$$y' = 6\, y + 1, \quad y(0) = 1.$$

1.1.10.- * Follow the steps below to solve

$$y' = -3y + 5, \quad y(0) = 1.$$

(a) Find any integrating factor μ for the differential equation.

(b) Write the differential equation as a total derivative of a potential function ψ.

(c) Use the potential function to find the general solution of the differential equation.

(d) Find the solution of the initial value problem above.

1.2. Linear Variable Coefficient Equations

In this section we obtain a formula for the solutions of variable coefficient linear equations, which generalizes Equation (1.1.4) in Theorem 1.1.2. To get this formula we use the integrating factor method—already used for constant coefficient equations in § 1.1. We also show that the initial value problem for variable coefficient equations has a unique solution—just as happens for constant coefficient equations.

In the last part of this section we turn our attention to a particular *nonlinear* differential equation—the Bernoulli equation. This nonlinear equation has a particular property: it can be transformed into a linear equation by an appropriate change in the unknown function. Then, one solves the linear equation for the changed function using the integrating factor method. The last step is to transform the changed function back into the original function.

1.2.1. Review: Constant Coefficient Equations. Let us recall how we solved the constant coefficient case. We wrote the equation $y' = a\,y + b$ as follows

$$y' = a\Big(y + \frac{b}{a}\Big).$$

The critical step was the following: since b/a is constant, then $(b/a)' = 0$, hence

$$\Big(y + \frac{b}{a}\Big)' = a\Big(y + \frac{b}{a}\Big).$$

At this point the equation was simple to solve,

$$\frac{(y + \frac{b}{a})'}{(y + \frac{a}{b})} = a \quad \Rightarrow \quad \ln\Big(\Big|y + \frac{b}{a}\Big|\Big)' = a \quad \Rightarrow \quad \ln\Big(\Big|y + \frac{b}{a}\Big|\Big) = c_0 + at.$$

We now computed the exponential on both sides, to get

$$\Big|y + \frac{b}{a}\Big| = e^{c_0 + at} = e^{c_0}\,e^{at} \quad \Rightarrow \quad y + \frac{b}{a} = (\pm e^{c_0})\,e^{at},$$

and calling $c = \pm e^{c_0}$ we got the formula

$$y(t) = c\,e^{at} - \frac{b}{a},$$

This idea can be generalized to variable coefficient equations, but only in the case where b/a is constant. For example, consider the case $b = 0$ and a depending on t. The equation is $y' = a(t)\,y$, and we can solve it as follows,

$$\frac{y'}{y} = a(t) \quad \Rightarrow \quad \ln(|y|)' = a(t) \quad \Rightarrow \quad \ln(|y(t)|) = A(t) + c_0,$$

where $A = \int a\,dt$, is a primitive or antiderivative of a. Therefore,

$$y(t) = \pm e^{A(t) + c_0} = \pm e^{A(t)}\,e^{c_0},$$

so we get the solution $y(t) = c\,e^{A(t)}$, where $c = \pm e^{c_0}$.

Example 1.2.1. The solutions of $y' = 2t\,y$ are $y(t) = c\,e^{t^2}$, where $c \in \mathbb{R}$.

However, the case where b/a is not constant is not so simple to solve—we cannot add zero to the equation in the form of $0 = (b/a)'$. We need a new idea. We now show an idea that works with all first order linear equations with variable coefficients—the integrating factor method.

1.2.2. Solving Variable Coefficient Equations.

We now state our main result—the formula for the solutions of linear differential equations with variable coefficiets.

Theorem 1.2.1 (Variable Coefficients)**.** *If the functions a, b are continuous, then*

$$y' = a(t)\, y + b(t), \tag{1.2.1}$$

has infinitely many solutions given by

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt, \tag{1.2.2}$$

where $A(t) = \int a(t)\, dt$ and $c \in \mathbb{R}$.

Remarks:

(a) The expression in Eq. (1.2.2) is called the *general solution* of the differential equation.
(b) The function $\mu(t) = e^{-A(t)}$ is called the *integrating factor* of the equation.

Example 1.2.2. Show that for constant coefficient equations the solution formula given in Eq. (1.2.2) reduces to Eq. (1.1.4).

Solution: In the particular case of constant coefficient equations, a primitive, or antiderivative, for the constant function a is $A(t) = at$, so

$$y(t) = c\, e^{at} + e^{at} \int e^{-at}\, b\, dt.$$

Since b is constant, the integral in the second term above can be computed explicitly,

$$e^{at} \int b\, e^{-at}\, dt = e^{at}\Big(-\frac{b}{a}\, e^{-at}\Big) = -\frac{b}{a}.$$

Therefore, in the case of a, b constants we obtain $y(t) = c\, e^{at} - \dfrac{b}{a}$ given in Eq. (1.1.4). ◁

Proof of Theorem 1.2.1: Write the differential equation with y on one side only,

$$y' - a\, y = b,$$

and then multiply the differential equation by a function μ, called an integrating factor,

$$\mu\, y' - a\, \mu\, y = \mu\, b. \tag{1.2.3}$$

The critical step is to choose a function μ such that

$$-\, a\, \mu = \mu'. \tag{1.2.4}$$

For any function μ solution of Eq. (1.2.4), the differential equation in (1.2.3) has the form

$$\mu\, y' + \mu'\, y = \mu\, b.$$

But the left-hand side is a total derivative of a product of two functions,

$$\big(\mu\, y\big)' = \mu\, b. \tag{1.2.5}$$

This is the property we want in an integrating factor, μ. We want to find a function μ such that the left-hand side of the differential equation for y can be written as a total derivative, just as in Eq. (1.2.5). We need to find just one of such functions μ. So we go back to Eq. (1.2.4), the differential equation for μ, which is simple to solve,

$$\mu' = -a\, \mu \quad \Rightarrow \quad \frac{\mu'}{\mu} = -a \quad \Rightarrow \quad \ln(|\mu|)' = -a \quad \Rightarrow \quad \ln(|\mu|) = -A + c_0,$$

where $A = \int a\,dt$, a primitive or antiderivative of a, and c_0 is an arbitrary constant. Computing the exponential of both sides we get

$$\mu = \pm e^{c_0}\, e^{-A} \quad \Rightarrow \quad \mu = c_1\, e^{-A}, \qquad c_1 = \pm e^{c_0}.$$

Since c_1 is a constant which will cancel out from Eq. (1.2.3) anyway, we choose the integration constant $c_0 = 0$, hence $c_1 = 1$. The integrating factor is then

$$\mu(t) = e^{-A(t)}.$$

This function is an integrating factor, because if we start again at Eq. (1.2.3), we get

$$e^{-A}\, y' - a\, e^{-A}\, y = e^{-A}\, b \quad \Rightarrow \quad e^{-A}\, y' + \big(e^{-A}\big)'\, y = e^{-A}\, b,$$

where we used the main property of the integrating factor, $-a\, e^{-A} = \big(e^{-A}\big)'$. Now the product rule for derivatives implies that the left-hand side above is a total derivative,

$$\big(e^{-A}\, y\big)' = e^{-A}\, b.$$

Integrating on both sides we get

$$\big(e^{-A}\, y\big) = \int e^{-A}\, b\, dt + c \quad \Rightarrow \quad \big(e^{-A}\, y\big) - \int e^{-A}\, b\, dt = c.$$

The function $\psi(t, y) = \big(e^{-A}\, y\big) - \int e^{-A}\, b\, dt$ is called a *potential function* of the differential equation. The solution of the differential equation can be computed form the second equation above, $\psi = c$, and the result is

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt.$$

This establishes the Theorem. □

Example 1.2.3. Find all solutions y to the differential equation

$$y' = \frac{3}{t}\, y + t^5, \qquad t > 0.$$

Solution: Rewrite the equation with y on only one side,

$$y' - \frac{3}{t}\, y = t^5.$$

Multiply the differential equation by a function μ, which we determine later,

$$\mu(t)\Big(y' - \frac{3}{t}\, y\Big) = t^5\, \mu(t) \quad \Rightarrow \quad \mu(t)\, y' - \frac{3}{t}\, \mu(t)\, y = t^5\, \mu(t).$$

We need to choose a positive function μ having the following property,

$$-\frac{3}{t}\, \mu(t) = \mu'(t) \quad \Rightarrow \quad -\frac{3}{t} = \frac{\mu'(t)}{\mu(t)} \quad \Rightarrow \quad -\frac{3}{t} = \big(\ln(|\mu|)\big)'$$

Integrating,

$$\ln(|\mu|) = -\int \frac{3}{t}\, dt = -3\, \ln(|t|) + c_0 = \ln(|t|^{-3}) + c_0 \quad \Rightarrow \quad \mu = (\pm e^{c_0})\, e^{\ln(|t|^{-3})},$$

so we get $\mu = (\pm e^{c_0})\, |t|^{-3}$. We need only one integrating factor, so we choose $\mu = t^{-3}$. We now go back to the differential equation for y and we multiply it by this integrating factor,

$$t^{-3}\Big(y' - \frac{3}{t}\, y\Big) = t^{-3}\, t^5 \quad \Rightarrow \quad t^{-3}\, y' - 3\, t^{-4}\, y = t^2.$$

Using that $-3\,t^{-4} = (t^{-3})'$ and $t^2 = \Big(\dfrac{t^3}{3}\Big)'$, we get

$$t^{-3}\,y' + (t^{-3})'\,y = \Big(\frac{t^3}{3}\Big)' \quad\Rightarrow\quad \big(t^{-3}\,y\big)' = \Big(\frac{t^3}{3}\Big)' \quad\Rightarrow\quad \Big(t^{-3}\,y - \frac{t^3}{3}\Big)' = 0.$$

This last equation is a total derivative of a potential function $\psi(t,y) = t^{-3}\,y - \dfrac{t^3}{3}$. Since the equation is a total derivative, this confirms that we got a correct integrating factor. Now we need to integrate the total derivative, which is simple to do,

$$t^{-3}\,y - \frac{t^3}{3} = c \quad\Rightarrow\quad t^{-3}\,y = c + \frac{t^3}{3} \quad\Rightarrow\quad y(t) = c\,t^3 + \frac{t^6}{3},$$

where c is an arbitrary constant. ◁

Example 1.2.4. Find all solutions of $ty' = -2y + 4t^2$, with $t > 0$.

Solution: Rewrite the equation as

$$y' = -\frac{2}{t}\,y + 4t \quad\Leftrightarrow\quad a(t) = -\frac{2}{t}, \quad b(t) = 4t. \tag{1.2.6}$$

Rewrite again,

$$y' + \frac{2}{t}\,y = 4t.$$

Multiply by a function μ,

$$\mu\,y' + \frac{2}{t}\,\mu\,y = \mu\,4t.$$

Choose μ solution of

$$\frac{2}{t}\,\mu = \mu' \quad\Rightarrow\quad \ln(|\mu|)' = \frac{2}{t} \quad\Rightarrow\quad \ln(|\mu|) = 2\ln(|t|) = \ln(t^2) \quad\Rightarrow\quad \mu(t) = \pm t^2.$$

We choose $\mu = t^2$. Multiply the differential equation by this μ,

$$t^2\,y' + 2t\,y = 4t\,t^2 \quad\Rightarrow\quad (t^2\,y)' = 4t^3.$$

If we write the right-hand side also as a derivative,

$$\big(t^2\,y\big)' = \big(t^4\big)' \quad\Rightarrow\quad \big(t^2\,y - t^4\big)' = 0.$$

So a potential function is $\psi(t,y(t)) = t^2\,y(t) - t^4$. Integrating on both sides we obtain

$$t^2\,y - t^4 = c \quad\Rightarrow\quad t^2\,y = c + t^4 \quad\Rightarrow\quad y(t) = \frac{c}{t^2} + t^2.$$

◁

1.2.3. The Initial Value Problem.

We now generalize Theorem 1.1.4—initial value problems have unique solutions—from constant coefficients to variable coefficients equations. We start introducing the initial value problem for a variable coefficients equation—a simple generalization of Def. 1.1.3.

Definition 1.2.2. *The* ***initial value problem*** *(IVP) is to find all solutions y of*

$$y' = a(t)\,y + b(t), \tag{1.2.7}$$

that satisfy the initial condition

$$y(t_0) = y_0, \tag{1.2.8}$$

where a, b are given functions and t_0, y_0 are given constants.

Remark: The Equation (1.2.8) is the *initial condition* of the problem.

Although the differential equation in (1.2.7) has infinitely many solutions, the associated initial value problem has a unique solution.

Theorem 1.2.3 (Variable coefficients IVP)**.** *Given continuous functions a, b, with domain (t_1, t_2), and constants $t_0 \in (t_1, t_2)$ and $y_0 \in \mathbb{R}$, the initial value problem*

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0, \tag{1.2.9}$$

has the unique solution y on the domain (t_1, t_2), given by

$$y(t) = y_0\, e^{A(t)} + e^{A(t)} \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds, \tag{1.2.10}$$

where the function $A(t) = \displaystyle\int_{t_0}^{t} a(s)\, ds$ is a particular antiderivative of function a.

Remark: In the particular case of a constant coefficient equation, where $a, b \in \mathbb{R}$, the solution given in Eq. (1.2.10) reduces to the one given in Eq. (1.1.12). Indeed,

$$A(t) = \int_{t_0}^{t} a\, ds = a(t - t_0), \qquad \int_{t_0}^{t} e^{-a(s-t_0)}\, b\, ds = -\frac{b}{a}\, e^{-a(t-t_0)} + \frac{b}{a}.$$

Therefore, the solution y can be written as

$$y(t) = y_0\, e^{a(t-t_0)} + e^{a(t-t_0)} \Big(-\frac{b}{a}\, e^{-a(t-t_0)} + \frac{b}{a}\Big) = \Big(y_0 + \frac{b}{a}\Big)\, e^{a(t-t_0)} - \frac{b}{a}.$$

Proof Theorem 1.2.3: Theorem 1.2.1 gives us the general solution of Eq. (1.2.9),

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt, \qquad c \in \mathbb{R}.$$

Let us use the notation $K(t) = \displaystyle\int e^{-A(t)}\, b(t)\, dt$, and then introduce the initial condition in (1.2.9), which fixes the constant c,

$$y_0 = y(t_0) = c\, e^{A(t_0)} + e^{A(t_0)} K(t_0).$$

So we get the constant c,

$$c = y_0\, e^{-A(t_0)} - K(t_0).$$

Using this expression in the general solution above,

$$y(t) = \Big(y_0\, e^{-A(t_0)} - K(t_0)\Big)\, e^{A(t)} + e^{A(t)}\, K(t) = y_0\, e^{A(t)-A(t_0)} + e^{A(t)} \big(K(t) - K(t_0)\big).$$

Let us introduce the particular primitives $\hat{A}(t) = A(t) - A(t_0)$ and $\hat{K}(t) = K(t) - K(t_0)$, which vanish at t_0, that is,

$$\hat{A}(t) = \int_{t_0}^{t} a(s)\, ds, \qquad \hat{K}(t) = \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds.$$

Then the solution y of the IVP has the form

$$y(t) = y_0\, e^{\hat{A}(t)} + e^{A(t)} \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds$$

which is equivalent to

$$y(t) = y_0\, e^{\hat{A}(t)} + e^{A(t)-A(t_0)} \int_{t_0}^{t} e^{-(A(s)-A(t_0))}\, b(s)\, ds,$$

so we conclude that

$$y(t) = y_0\, e^{\hat{A}(t)} + e^{\hat{A}(t)} \int_{t_0}^{t} e^{-\hat{A}(s)}\, b(s)\, ds.$$

Once we rename the particular primitive $\hat{A}$ simply by A, we establish the Theorem. $\square$

We solve the next Example following the main steps in the proof of Theorem 1.2.3 above.

Example 1.2.5. Find the function y solution of the initial value problem

$$ty' + 2y = 4t^2, \qquad t > 0, \qquad y(1) = 2.$$

Solution: In Example 1.2.4 we computed the general solution of the differential equation,

$$y(t) = \frac{c}{t^2} + t^2, \qquad c \in \mathbb{R}.$$

The initial condition implies that

$$2 = y(1) = c + 1 \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y(t) = \frac{1}{t^2} + t^2.$$

$\lhd$

Example 1.2.6. Find the solution of the problem given in Example 1.2.5, but this time using the results of Theorem 1.2.3.

Solution: We find the solution simply by using Eq. (1.2.10). First, find the integrating factor function μ as follows:

$$A(t) = -\int_1^t \frac{2}{s}\, ds = -2\big[\ln(t) - \ln(1)\big] = -2\ln(t) \quad \Rightarrow \quad A(t) = \ln(t^{-2}).$$

The integrating factor is $\mu(t) = e^{-A(t)}$, that is,

$$\mu(t) = e^{-\ln(t^{-2})} = e^{\ln(t^2)} \quad \Rightarrow \quad \mu(t) = t^2.$$

Note that Eq. (1.2.10) contains $e^{A(t)} = 1/\mu(t)$. Then, compute the solution as follows,

$$\begin{aligned} y(t) &= \frac{1}{t^2}\Big(2 + \int_1^t s^2\, 4s\, ds\Big) \\ &= \frac{2}{t^2} + \frac{1}{t^2}\int_1^t 4s^3 ds \\ &= \frac{2}{t^2} + \frac{1}{t^2}(t^4 - 1) \\ &= \frac{2}{t^2} + t^2 - \frac{1}{t^2} \quad \Rightarrow \quad y(t) = \frac{1}{t^2} + t^2. \end{aligned}$$

$\lhd$

1.2.4. The Bernoulli Equation. In 1696 Jacob Bernoulli solved what is now known as the Bernoulli differential equation. This is a first order *nonlinear* differential equation. The following year Leibniz solved this equation by transforming it into a linear equation. We now explain Leibniz's idea in more detail.

Definition 1.2.4. *The* ***Bernoulli equation*** *is*

$$y' = p(t)\, y + q(t)\, y^n. \tag{1.2.11}$$

where p, q are given functions and $n \in \mathbb{R}$.

Remarks:

(a) For $n \neq 0, 1$ the equation is nonlinear.

(b) If $n = 2$ we get the *logistic equation*, (we'll study it in a later chapter),

$$y' = ry\left(1 - \frac{y}{K}\right).$$

(c) This is not the Bernoulli equation from fluid dynamics.

The Bernoulli equation is special in the following sense: it is a nonlinear equation that can be transformed into a linear equation.

Theorem 1.2.5 (Bernoulli)**.** *The function y is a solution of the Bernoulli equation*

$$y' = p(t)\, y + q(t)\, y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' = -(n-1)p(t)\, v - (n-1)q(t).$$

Remark: This result summarizes Laplace's idea to solve the Bernoulli equation. To transform the Bernoulli equation for y, which is nonlinear, into a linear equation for $v = 1/y^{(n-1)}$. One then solves the linear equation for v using the integrating factor method. The last step is to transform back to $y = (1/v)^{1/(n-1)}$.

Proof of Theorem 1.2.5: Divide the Bernoulli equation by y^n,

$$\frac{y'}{y^n} = \frac{p(t)}{y^{n-1}} + q(t).$$

Introduce the new unknown $v = y^{-(n-1)}$ and compute its derivative,

$$v' = \left[y^{-(n-1)}\right]' = -(n-1)y^{-n}\, y' \quad \Rightarrow \quad -\frac{v'(t)}{(n-1)} = \frac{y'(t)}{y^n(t)}.$$

If we substitute v and this last equation into the Bernoulli equation we get

$$-\frac{v'}{(n-1)} = p(t)\, v + q(t) \quad \Rightarrow \quad v' = -(n-1)p(t)\, v - (n-1)q(t).$$

This establishes the Theorem. □

Example 1.2.7. Find every nonzero solution of the differential equation

$$y' = y + 2\, y^5.$$

Solution: This is a Bernoulli equation for $n = 5$. Divide the equation by y^5,

$$\frac{y'}{y^5} = \frac{1}{y^4} + 2.$$

Introduce the function $v = 1/y^4$ and its derivative $v' = -4(y'/y^5)$, into the differential equation above,

$$-\frac{v'}{4} = v + 2 \quad \Rightarrow \quad v' = -4\, v - 8 \quad \Rightarrow \quad v' + 4\, v = -8.$$

The last equation is a linear differential equation for the function v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{4t}$, then

$$\left(e^{4t} v\right)' = -8\, e^{4t} \quad \Rightarrow \quad e^{4t} v = -\frac{8}{4}\, e^{4t} + c.$$

We obtain that $v = c\,e^{-4t} - 2$. Since $v = 1/y^4$,

$$\frac{1}{y^4} = c\,e^{-4t} - 2 \quad \Rightarrow \quad y(t) = \pm\frac{1}{\big(c\,e^{-4t} - 2\big)^{1/4}}.$$

$\triangleleft$

Example 1.2.8. Given any constants a_0, b_0, find every solution of the differential equation

$$y' = a_0\,y + b_0\,y^3.$$

Solution: This is a Bernoulli equation with $n = 3$. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{a_0}{y^2} + b_0.$$

Introduce the function $v = 1/y^2$ and its derivative $v' = -2(y'/y^3)$, into the differential equation above,

$$-\frac{v'}{2} = a_0 v + b_0 \quad \Rightarrow \quad v' = -2a_0 v - 2b_0 \quad \Rightarrow \quad v' + 2a_0 v = -2b_0.$$

The last equation is a linear differential equation for v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{2a_0 t}$,

$$\big(e^{2a_0 t} v\big)' = -2b_0\,e^{2a_0 t} \quad \Rightarrow \quad e^{2a_0 t} v = -\frac{b_0}{a_0}\,e^{2a_0 t} + c$$

We obtain that $v = c\,e^{-2a_0 t} - \dfrac{b_0}{a_0}$. Since $v = 1/y^2$,

$$\frac{1}{y^2} = c\,e^{-2a_0 t} - \frac{b_0}{a_0} \quad \Rightarrow \quad y(t) = \pm\frac{1}{\big(c\,e^{-2a_0 t} - \frac{b_0}{a_0}\big)^{1/2}}.$$

$\triangleleft$

Example 1.2.9. Find every solution of the equation $t\,y' = 3y + t^5\,y^{1/3}$.

Solution: Rewrite the differential equation as

$$y' = \frac{3}{t}\,y + t^4\,y^{1/3}.$$

This is a Bernoulli equation for $n = 1/3$. Divide the equation by $y^{1/3}$,

$$\frac{y'}{y^{1/3}} = \frac{3}{t}\,y^{2/3} + t^4.$$

Define the new unknown function $v = 1/y^{(n-1)}$, that is, $v = y^{2/3}$, compute is derivative, $v' = \dfrac{2}{3}\,\dfrac{y'}{y^{1/3}}$, and introduce them in the differential equation,

$$\frac{3}{2}\,v' = \frac{3}{t}\,v + t^4 \quad \Rightarrow \quad v' - \frac{2}{t}\,v = \frac{2}{3}\,t^4.$$

This is a linear equation for v. Integrate this equation using the integrating factor method. To compute the integrating factor we need to find

$$A(t) = \int \frac{2}{t}\,dt = 2\ln(t) = \ln(t^2).$$

Then, the integrating factor is $\mu(t) = e^{-A(t)}$. In this case we get

$$\mu(t) = e^{-\ln(t^2)} = e^{\ln(t^{-2})} \quad \Rightarrow \quad \mu(t) = \frac{1}{t^2}.$$

Therefore, the equation for v can be written as a total derivative,

$$\frac{1}{t^2}\big(v' - \frac{2}{t}\,v\big) = \frac{2}{3}\,t^2 \quad \Rightarrow \quad \Big(\frac{v}{t^2} - \frac{2}{9}\,t^3\Big)' = 0.$$

The potential function is $\psi(t,v) = v/t^2 - (2/9)t^3$ and the solution of the differential equation is $\psi(t, v(t)) = c$, that is,

$$\frac{v}{t^2} - \frac{2}{9}\,t^3 = c \quad \Rightarrow \quad v(t) = t^2\left(c + \frac{2}{9}\,t^3\right) \quad \Rightarrow \quad v(t) = c\,t^2 + \frac{2}{9}\,t^5.$$

Once v is known we compute the original unknown $y = \pm v^{3/2}$, where the double sign is related to taking the square root. We finally obtain

$$y(t) = \pm\Big(c\,t^2 + \frac{2}{9}\,t^5\Big)^{3/2}.$$

◁

Notes. This section corresponds to Boyce-DiPrima [3] Section 2.1, and Simmons [10] Section 2.10. The Bernoulli equation is solved in the exercises of section 2.4 in Boyce-Diprima, and in the exercises of section 2.10 in Simmons.

1.2.5. Exercises.

1.2.1.- Find all solutions of
$$y' = 4t\, y.$$

1.2.2.- Find the general solution of
$$y' = -y + e^{-2t}.$$

1.2.3.- Find the solution y to the IVP
$$y' = y + 2te^{2t}, \quad y(0) = 0.$$

1.2.4.- Find the solution y to the IVP
$$t\, y' + 2\, y = \frac{\sin(t)}{t}, \quad y\Big(\frac{\pi}{2}\Big) = \frac{2}{\pi},$$
for $t > 0$.

1.2.5.- Find all solutions y to the ODE
$$\frac{y'}{(t^2+1)y} = 4t.$$

1.2.6.- Find all solutions y to the ODE
$$ty' + n\, y = t^2,$$
with n a positive integer.

1.2.7.- Find the solutions to the IVP
$$2ty - y' = 0, \quad y(0) = 3.$$

1.2.8.- Find all solutions of the equation
$$y' = y - 2\sin(t).$$

1.2.9.- Find the solution to the initial value problem
$$t\, y' = 2\, y + 4t^3\cos(4t), \quad y\ \frac{\pi}{8}\Big) = 0.$$

1.2.10.- Find all solutions of the equation
$$y' + t\, y = t\, y^2.$$

1.2.11.- Find all solutions of the equation
$$y' = -x\, y + 6x\sqrt{y}.$$

1.2.12.- Find all solutions of the IVP
$$y' = y + \frac{3}{y^2}, \quad y(0) = 1.$$

1.2.13.- * Find all solutions of
$$y' = a\, y + b\, y^n,$$
where $a \neq 0$, b, and n are real constants with $n \neq 0, 1$.

1.3. Separable Equations

1.3.1. Separable Equations. More often than not nonlinear differential equations are harder to solve than linear equations. Separable equations are an exception—they can be solved just by integrating on both sides of the differential equation. We tried this idea to solve linear equations, but it did not work. However, it works for separable equations.

Definition 1.3.1. *A **separable** differential equation for the function y is*

$$h(y)\,y' = g(t),$$

where h, g are given functions.

Remark: A separable differential equation is $h(y)\,y' = g(y)$ has the following properties:

- The left-hand side depends explicitly only on y, so any t dependence is through y.
- The right-hand side depends only on t.
- And the left-hand side is of the form (something on y) × y'.

Example 1.3.1.

(a) The differential equation $y' = \dfrac{t^2}{1-y^2}$ is separable, since it is equivalent to

$$\left(1-y^2\right) y' = t^2 \quad \Rightarrow \quad \begin{cases} g(t) = t^2, \\ h(y) = 1-y^2. \end{cases}$$

(b) The differential equation $y' + y^2\,\cos(2t) = 0$ is separable, since it is equivalent to

$$\frac{1}{y^2}\,y' = -\cos(2t) \quad \Rightarrow \quad \begin{cases} g(t) = -\cos(2t), \\ h(y) = \dfrac{1}{y^2}. \end{cases}$$

The functions g and h are not uniquely defined; another choice in this example is:

$$g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}.$$

(c) The linear differential equation $y' = a(t)\,y$ is separable, since it is equivalent to

$$\frac{1}{y}\,y' = a(t) \quad \Rightarrow \quad \begin{cases} g(t) = a(t), \\ h(y) = \dfrac{1}{y}. \end{cases}$$

(d) The equation $y' = e^y + \cos(t)$ is not separable.

(e) The constant coefficient linear differential equation $y' = a_0\,y + b_0$ is separable, since it is equivalent to

$$\frac{1}{(a_0\,y + b_0)}\,y' = 1 \quad \Rightarrow \quad \begin{cases} g(t) = 1, \\ h(y) = \dfrac{1}{(a_0\,y + b_0)}. \end{cases}$$

(f) The linear equation $y' = a(t)\,y + b(t)$, with $a \neq 0$ and b/a nonconstant, is not separable.

◁

From the last two examples above we see that linear differential equations, with $a \neq 0$, are separable for b/a constant, and not separable otherwise. Separable differential equations are simple to solve. We just integrate on both sides of the equation. We show this idea in the following example.

Example 1.3.2. Find all solutions y to the differential equation

$$-\frac{y'}{y^2} = \cos(2t).$$

Solution: The differential equation above is separable, with

$$g(t) = \cos(2t), \qquad h(y) = -\frac{1}{y^2}.$$

Therefore, it can be integrated as follows:

$$-\frac{y'}{y^2} = \cos(2t) \quad \Leftrightarrow \quad \int -\frac{y'(t)}{y^2(t)}\,dt = \int \cos(2t)\,dt + c.$$

The integral on the right-hand side can be computed explicitly. The integral on the left-hand side can be done by substitution. The substitution is

$$u = y(t), \qquad du = y'(t)\,dt \quad \Rightarrow \quad \int -\frac{du}{u^2}\,dt = \int \cos(2t)\,dt + c.$$

This notation makes clear that u is the new integation variable, while $y(t)$ are the unknown function values we look for. However it is common in the literature to use the same name for the variable and the unknown function. We will follow that convention, and we write the substitution as

$$y = y(t), \qquad dy = y'(t)\,dt \quad \Rightarrow \quad \int -\frac{dy}{y^2}\,dt = \int \cos(2t)\,dt + c.$$

We hope this is not too confusing. Integrating on both sides above we get

$$\frac{1}{y} = \frac{1}{2}\sin(2t) + c.$$

So, we get the implicit and explicit form of the solution,

$$\frac{1}{y(t)} = \frac{1}{2}\sin(2t) + c \quad \Leftrightarrow \quad y(t) = \frac{2}{\sin(2t) + 2c}.$$

$\lhd$

Remark: Notice the following about the equation and its implicit solution:

$$-\frac{1}{y^2}\,y' = \cos(2t) \quad \Leftrightarrow \quad h(y)\,y' = g(t), \qquad h(y) = -\frac{1}{y^2}, \qquad g(t) = \cos(2t),$$

$$\frac{1}{y}\,y' = \frac{1}{2}\sin(2t) \quad \Leftrightarrow \quad H(y) = G(t), \qquad H(y) = \frac{1}{y}, \qquad G(t) = \frac{1}{2}\sin(2t).$$

- Here H is an antiderivative of h, that is, $H(y) = \int h(y)\,dy$.
- Here G is an antiderivative of g, that is, $G(t) = \int g(t)\,dt$.

Theorem 1.3.2 (Separable Equations). *If h, g are continuous, with $h \neq 0$, then*

$$h(y)\,y' = g(t) \tag{1.3.1}$$

has infinitely many solutions y satisfying the algebraic equation

$$H(y(t)) = G(t) + c, \tag{1.3.2}$$

where $c \in \mathbb{R}$ is arbitrary, H and G are antiderivatives of h and g.

Remark: An antiderivative of h is $H(y) = \int h(y)\,dy$, while an antiderivative of g is the function $G(t) = \int g(t)\,dt$.

Proof of Theorem 1.3.2: Integrate with respect to t on both sides in Eq. (1.3.1),

$$h(y)\,y' = g(t) \quad \Rightarrow \quad \int h(y(t))\,y'(t)\,dt = \int g(t)\,dt + c,$$

where c is an arbitrary constant. Introduce on the left-hand side of the second equation above the substitution

$$y = y(t), \qquad dy = y'(t)\,dt.$$

The result of the substitution is

$$\int h(y(t))\,y'(t)\,dt = \int h(y)dy \quad \Rightarrow \quad \int h(y)\,dy = \int g(t)\,dt + c.$$

To integrate on each side of this equation means to find a function H, primitive of h, and a function G, primitive of g. Using this notation we write

$$H(y) = \int h(y)\,dy, \qquad G(t) = \int g(t)\,dt.$$

Then the equation above can be written as follows,

$$H(y) = G(t) + c,$$

which implicitly defines a function y, which depends on t. This establishes the Theorem. $\square$

Example 1.3.3. Find all solutions y to the differential equation

$$y' = \frac{t^2}{1 - y^2}. \tag{1.3.3}$$

Solution: We write the differential equation in (1.3.3) in the form $h(y)\,y' = g(t)$,

$$\big(1 - y^2\big)\,y' = t^2.$$

In this example the functions h and g defined in Theorem 1.3.2 are given by

$$h(y) = (1 - y^2), \qquad g(t) = t^2.$$

We now integrate with respect to t on both sides of the differential equation,

$$\int \big(1 - y^2(t)\big)\,y'(t)\,dt = \int t^2\,dt + c,$$

where c is any constant. The integral on the right-hand side can be computed explicitly. The integral on the left-hand side can be done by substitution. The substitution is

$$y = y(t), \qquad dy = y'(t)\,dt \quad \Rightarrow \quad \int \big(1 - y^2(t)\big)\,y'(t)\,dt = \int (1 - y^2)\,dy.$$

This substitution on the left-hand side integral above gives,

$$\int (1 - y^2)\,dy = \int t^2\,dt + c \quad \Leftrightarrow \quad y - \frac{y^3}{3} = \frac{t^3}{3} + c.$$

The equation above defines a function y, which depends on t. We can write it as

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c.$$

We have solved the differential equation, since there are no derivatives in the last equation. When the solution is given in terms of an algebraic equation, we say that the solution y is given in *implicit form*. ◁

Definition 1.3.3. *A function y is a solution in* ***implicit form*** *of the equation $h(y)\,y' = g(t)$ iff the function y is solution of the algebraic equation*

$$H\big(y(t)\big) = G(t) + c,$$

where H and G are any antiderivatives of h and g. In the case that function H is invertible, the solution y above is given in ***explicit form*** *iff is written as*

$$y(t) = H^{-1}\big(G(t) + c\big).$$

In the case that H is not invertible or H^{-1} is difficult to compute, we leave the solution y in implicit form. We now solve the same example as in Example 1.3.3, but now we just use the result of Theorem 1.3.2.

Example 1.3.4. Use the formula in Theorem 1.3.2 to find all solutions y to the equation

$$y' = \frac{t^2}{1 - y^2}. \tag{1.3.4}$$

Solution: Theorem 1.3.2 tell us how to obtain the solution y. Writing Eq. (1.3.4) as

$$\big(1 - y^2\big)\,y' = t^2,$$

we see that the functions h, g are given by

$$h(y) = 1 - y^2, \qquad g(t) = t^2.$$

Their primitive functions, H and G, respectively, are simple to compute,

$$h(y) = 1 - y^2 \quad \Rightarrow \quad H(y) = y - \frac{y^3}{3},$$
$$g(t) = t^2 \quad \Rightarrow \quad G(t) = \frac{t^3}{3}.$$

Then, Theorem 1.3.2 implies that the solution y satisfies the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \tag{1.3.5}$$

where $c \in \mathbb{R}$ is arbitrary. ◁

Remark: Sometimes it is simpler to remember ideas than formulas. So one can solve a separable equation as we did in Example 1.3.3, instead of using the solution formulas, as in Example 1.3.4. (Although in the case of separable equations both methods are very close.)

In the next Example we show that an initial value problem can be solved even when the solutions of the differential equation are given in implicit form.

Example 1.3.5. Find the solution of the initial value problem

$$y' = \frac{t^2}{1 - y^2}, \qquad y(0) = 1. \tag{1.3.6}$$

Solution: From Example 1.3.3 we know that all solutions to the differential equation in (1.3.6) are given by

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c,$$

where $c \in \mathbb{R}$ is arbitrary. This constant c is now fixed with the initial condition in Eq. (1.3.6)

$$y(0) - \frac{y^3(0)}{3} = \frac{0}{3} + c \quad \Rightarrow \quad 1 - \frac{1}{3} = c \quad \Leftrightarrow \quad c = \frac{2}{3} \quad \Rightarrow \quad y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + \frac{2}{3}.$$

So we can rewrite the algebraic equation defining the solution functions y as the (time dependent) roots of a cubic (in y) polynomial,

$$y^3(t) - 3y(t) + t^3 + 2 = 0.$$

◁

Example 1.3.6. Find the solution of the initial value problem

$$y' + y^2\cos(2t) = 0, \qquad y(0) = 1. \tag{1.3.7}$$

Solution: The differential equation above can be written as

$$-\frac{1}{y^2}\,y' = \cos(2t).$$

We know, from Example 1.3.2, that the solutions of the differential equation are

$$y(t) = \frac{2}{\sin(2t) + 2c}, \qquad c \in \mathbb{R}.$$

The initial condition implies that

$$1 = y(0) = \frac{2}{0 + 2c} \quad \Leftrightarrow \quad c = 1.$$

So, the solution to the IVP is given in explicit form by

$$y(t) = \frac{2}{\sin(2t) + 2}.$$

◁

Example 1.3.7. Follow the proof in Theorem 1.3.2 to find all solutions y of the equation

$$y' = \frac{4t - t^3}{4 + y^3}.$$

Solution: The differential equation above is separable, with

$$h(y) = 4 + y^3, \qquad g(t) = 4t - t^3.$$

Therefore, it can be integrated as follows:

$$\big(4 + y^3\big)\,y' = 4t - t^3 \quad \Leftrightarrow \quad \int \big(4 + y^3(t)\big)\,y'(t)\,dt = \int (4t - t^3)\,dt + c.$$

Again the substitution

$$y = y(t), \qquad dy = y'(t)\,dt$$

implies that

$$\int (4 + y^3)\,dy = \int (4t - t^3)\,dt + c_0. \quad \Leftrightarrow \quad 4y + \frac{y^4}{4} = 2t^2 - \frac{t^4}{4} + c_0.$$

Calling $c_1 = 4c_0$ we obtain the following implicit form for the solution,

$$y^4(t) + 16y(t) - 8t^2 + t^4 = c_1.$$

◁

Example 1.3.8. Find the solution of the initial value problem below in explicit form,

$$y' = \frac{2-t}{1+y}, \qquad y(0) = 1. \tag{1.3.8}$$

Solution: The differential equation above is separable with

$$h(y) = 1 + y, \qquad g(t) = 2 - t.$$

Their primitives are respectively given by,

$$\begin{aligned} h(y) = 1 + y \quad &\Rightarrow \quad H(y) = y + \frac{y^2}{2}, \\ g(t) = 2 - t \quad &\Rightarrow \quad G(t) = 2t - \frac{t^2}{2}. \end{aligned}$$

Therefore, the implicit form of all solutions y to the ODE above are given by

$$y(t) + \frac{y^2(t)}{2} = 2t - \frac{t^2}{2} + c,$$

with $c \in \mathbb{R}$. The initial condition in Eq. (1.3.8) fixes the value of constant c, as follows,

$$y(0) + \frac{y^2(0)}{2} = 0 + c \quad \Rightarrow \quad 1 + \frac{1}{2} = c \quad \Rightarrow \quad c = \frac{3}{2}.$$

We conclude that the implicit form of the solution y is given by

$$y(t) + \frac{y^2(t)}{2} = 2t - \frac{t^2}{2} + \frac{3}{2}, \quad \Leftrightarrow \quad y^2(t) + 2y(t) + (t^2 - 4t - 3) = 0.$$

The explicit form of the solution can be obtained realizing that $y(t)$ is a root in the quadratic polynomial above. The two roots of that polynomial are given by

$$y_\pm(t) = \frac{1}{2}\big[-2 \pm \sqrt{4 - 4(t^2 - 4t - 3)}\big] \quad \Leftrightarrow \quad y_\pm(t) = -1 \pm \sqrt{-t^2 + 4t + 4}.$$

We have obtained two functions y_+ and y_-. However, we know that there is only one solution to the initial value problem. We can decide which one is the solution by evaluating them at the value $t = 0$ given in the initial condition. We obtain

$$\begin{aligned} y_+(0) &= -1 + \sqrt{4} = 1, \\ y_-(0) &= -1 - \sqrt{4} = -3. \end{aligned}$$

Therefore, the solution is y_+, that is, the explicit form of the solution is

$$y(t) = -1 + \sqrt{-t^2 + 4t + 4}.$$

◁

1.3.2. Euler Homogeneous Equations.

Sometimes a differential equation is not separable but it can be transformed into a separable equation changing the unknown function. This is the case for differential equations known as Euler homogenous equations.

Definition 1.3.4. *An* ***Euler homogeneous*** *differential equation has the form*

$$y'(t) = F\Big(\frac{y(t)}{t}\Big).$$

Remark:

(a) Any function F of t, y that depends only on the quotient y/t is *scale invariant*. This means that F does not change when we do the transformation $y \to cy$, $t \to ct$,

$$F\Big(\frac{(cy)}{(ct)}\Big) = F\Big(\frac{y}{t}\Big).$$

For this reason the differential equations above are also called *scale invariant* equations.

(b) Scale invariant functions are a particular case of *homogeneous functions of degree n*, which are functions f satisfying

$$f(ct, cy) = c^n\, f(y, t).$$

Scale invariant functions are the case $n = 0$.

(c) An example of an homogeneous function is the energy of a thermodynamical system, such as a gas in a bottle. The energy, E, of a fixed amount of gas is a function of the gas entropy, S, and the gas volume, V. Such energy is an homogeneous function of degree one,

$$E(cS, cV) = c\, E(S, V), \qquad \text{for all } c \in \mathbb{R}.$$

Example 1.3.9. Show that the functions f_1 and f_2 are homogeneous and find their degree,

$$f_1(t, y) = t^4y^2 + ty^5 + t^3y^3, \qquad f_2(t, y) = t^2y^2 + ty^3.$$

Solution: The function f_1 is homogeneous of degree 6, since

$$f_1(ct, cy) = c^4t^4\, c^2y^2 + ct\, c^5y^5 + c^3t^3\, c^3y^3 = c^6\, (t^4y^2 + ty^5 + t^3y^3) = c^6\, f(t, y).$$

Notice that the sum of the powers of t and y on every term is 6. Analogously, function f_2 is homogeneous degree 4, since

$$f_2(ct, cy) = c^2t^2\, c^2y^2 + ct\, c^3y^3 = c^4\, (t^2y^2 + ty^3) = c^4\, f_2(t, y).$$

And the sum of the powers of t and y on every term is 4. ◁

Example 1.3.10. Show that the functions below are scale invariant functions,

$$f_1(t, y) = \frac{y}{t}, \qquad f_2(t, y) = \frac{t^3 + t^2\, y + t\, y^2 + y^3}{t^3 + t\, y^2}.$$

Solution: Function f_1 is scale invariant since

$$f(ct, cy) = \frac{cy}{ct} = \frac{y}{t} = f(t, y).$$

The function f_2 is scale invariant as well, since

$$f_2(ct, cy) = \frac{c^3t^3 + c^2t^2\, cy + ct\, c^2y^2 + c^3y^3}{c^3t^3 + ct\, c^2y^2} = \frac{c^3(t^3 + t^2\, y + t\, y^2 + y^3)}{c^3(t^3 + t\, y^2)} = f_2(t, y).$$

◁

More often than not, Euler homogeneous differential equations come from a differential equation $N\, y' + M = 0$ where both N and M are homogeneous functions of the same degree.

Theorem 1.3.5. *If the functions N, M, of t, y, are homogeneous of the same degree, then the differential equation*

$$N(t,y)\, y'(t) + M(t,y) = 0$$

is Euler homogeneous.

Proof of Theorem 1.3.5: Rewrite the equation as

$$y'(t) = -\frac{M(t,y)}{N(t,y)},$$

The function $f(y,y) = -\dfrac{M(t,y)}{N(t,y)}$ is scale invariant, because

$$f(ct,cy) = -\frac{M(ct,cy)}{N(ct,cy)} = -\frac{c^n\, M(t,y)}{c^n\, N(t,y)} = -\frac{M(t,y)}{N(t,y)} = f(t,y),$$

where we used that M and N are homogeneous of the same degree n. We now find a function F such that the differential equation can be written as

$$y' = F\Big(\frac{y}{t}\Big).$$

Since M and N are homogeneous degree n, we multiply the differential equation by "1" in the form $(1/t)^n/(1/t)^n$, as follows,

$$y'(t) = -\frac{M(t,y)}{N(t,y)}\,\frac{(1/t^n)}{(1/t^n)} = -\frac{M((t/t),(y/t))}{N((t/t),(y/t))} = -\frac{M(1,(y/t))}{N(1,(y/t))} \quad\Rightarrow\quad y' = F\Big(\frac{y}{t}\Big),$$

where

$$F\Big(\frac{y}{t}\Big) = -\frac{M(1,(y/t))}{N(1,(y/t))}.$$

This establishes the Theorem. □

Example 1.3.11. Show that $(t-y)\, y' - 2y + 3t + \dfrac{y^2}{t} = 0$ is an Euler homogeneous equation.

Solution: Rewrite the equation in the standard form

$$(t-y)\, y' = 2y - 3t - \frac{y^2}{t} \quad\Rightarrow\quad y' = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t-y)}.$$

So the function f in this case is given by

$$f(t,y) = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t-y)}.$$

This function is scale invariant, since numerator and denominator are homogeneous of the same degree, $n = 1$ in this case,

$$f(ct,cy) = \frac{\Big(2cy - 3ct - \dfrac{c^2y^2}{ct}\Big)}{(ct-cy)} = \frac{c\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{c(t-y)} = f(t,y).$$

So, the differential equation is Euler homogeneous. We now write the equation in the form $y' = F(y/t)$. Since the numerator and denominator are homogeneous of degree $n = 1$, we multiply them by "1" in the form $(1/t)^1/(1/t)^1$, that is

$$y' = \frac{\left(2y - 3t - \dfrac{y^2}{t}\right)}{(t-y)} \, \frac{(1/t)}{(1/t)}.$$

Distribute the factors $(1/t)$ in numerator and denominator, and we get

$$y' = \frac{\big(2(y/t) - 3 - (y/t)^2\big)}{\big(1 - (y/t)\big)} \quad \Rightarrow \quad y' = F\Big(\frac{y}{t}\Big),$$

where

$$F\Big(\frac{y}{t}\Big) = \frac{\big(2(y/t) - 3 - (y/t)^2\big)}{\big(1 - (y/t)\big)}.$$

So, the equation is Euler homogeneous and it is written in the standard form. $\lhd$

Example 1.3.12. Determine whether the equation $(1 - y^3)\, y' = t^2$ is Euler homogeneous.

Solution: If we write the differential equation in the standard form, $y' = f(t, y)$, then we get $f(t, y) = \dfrac{t^2}{1 - y^3}$. But

$$f(ct, cy) = \frac{c^2 t^2}{1 - c^3 y^3} \neq f(t, y),$$

hence the equation is not Euler homogeneous. $\lhd$

1.3.3. Solving Euler Homogeneous Equations.

In § 1.2 we transformed a Bernoulli equation into an equation we knew how to solve, a linear equation. Theorem 1.3.6 transforms an Euler homogeneous equation into a separable equation, which we know how to solve.

Theorem 1.3.6. *The Euler homogeneous equation*

$$y' = F\Big(\frac{y}{t}\Big)$$

for the function y determines a separable equation for $v = y/t$, given by

$$\frac{v'}{\big(F(v) - v\big)} = \frac{1}{t}.$$

Remark: The original homogeneous equation for the function y is transformed into a separable equation for the unknown function $v = y/t$. One solves for v, in implicit or explicit form, and then transforms back to $y = t\, v$.

Proof of Theorem 1.3.6: Introduce the function $v = y/t$ into the differential equation,

$$y' = F(v).$$

We still need to replace y' in terms of v. This is done as follows,

$$y(t) = t\, v(t) \quad \Rightarrow \quad y'(t) = v(t) + t\, v'(t).$$

Introducing these expressions into the differential equation for y we get

$$v + t\, v' = F(v) \quad \Rightarrow \quad v' = \frac{\big(F(v) - v\big)}{t} \quad \Rightarrow \quad \frac{v'}{\big(F(v) - v\big)} = \frac{1}{t}.$$

The equation on the far right is separable. This establishes the Theorem. $\square$

Example 1.3.13. Find all solutions y of the differential equation $y' = \dfrac{t^2 + 3y^2}{2ty}$.

Solution: The equation is Euler homogeneous, since

$$f(ct, cy) = \frac{c^2t^2 + 3c^2y^2}{2(ct)(cy)} = \frac{c^2(t^2 + 3y^2)}{c^2(2ty)} = \frac{t^2 + 3y^2}{2ty} = f(t, y).$$

Next we compute the function F. Since the numerator and denominator are homogeneous degree "2" we multiply the right-hand side of the equation by "1" in the form $(1/t^2)/(1/t^2)$,

$$y' = \frac{(t^2 + 3y^2)}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Now we introduce the change of functions $v = y/t$,

$$y' = \frac{1 + 3v^2}{2v}.$$

Since $y = t\,v$, then $y' = v + t\,v'$, which implies

$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v} = \frac{1 + v^2}{2v}.$$

We obtained the separable equation

$$v' = \frac{1}{t}\left(\frac{1 + v^2}{2v}\right).$$

We rewrite and integrate it,

$$\frac{2v}{1 + v^2}\,v' = \frac{1}{t} \quad \Rightarrow \quad \int \frac{2v}{1 + v^2}\,v'\,dt = \int \frac{1}{t}\,dt + c_0.$$

The substitution $u = 1 + v^2(t)$ implies $du = 2v(t)\,v'(t)\,dt$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \quad \Rightarrow \quad \ln(u) = \ln(t) + c_0 \quad \Rightarrow \quad u = e^{\ln(t) + c_0}.$$

But $u = e^{\ln(t)}e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. So, we get

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left(\frac{y}{t}\right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\,\sqrt{c_1 t - 1}.$$

$\triangleleft$

Example 1.3.14. Find all solutions y of the differential equation $y' = \dfrac{t(y + 1) + (y + 1)^2}{t^2}$.

Solution: This equation is Euler homogeneous when written in terms of the unknown $u(t) = y(t) + 1$ and the variable t. Indeed, $u' = y'$, thus we obtain

$$y' = \frac{t(y + 1) + (y + 1)^2}{t^2} \quad \Leftrightarrow \quad u' = \frac{tu + u^2}{t^2} \quad \Leftrightarrow \quad u' = \frac{u}{t} + \left(\frac{u}{t}\right)^2.$$

Therefore, we introduce the new variable $v = u/t$, which satisfies $u = t\,v$ and $u' = v + t\,v'$. The differential equation for v is

$$v + t\,v' = v + v^2 \quad \Leftrightarrow \quad t\,v' = v^2 \quad \Leftrightarrow \quad \int \frac{v'}{v^2}\,dt = \int \frac{1}{t}\,dt + c,$$

with $c \in \mathbb{R}$. The substitution $w = v(t)$ implies $dw = v'\,dt$, so

$$\int w^{-2}\,dw = \int \frac{1}{t}\,dt + c \quad \Leftrightarrow \quad -w^{-1} = \ln(|t|) + c \quad \Leftrightarrow \quad w = -\frac{1}{\ln(|t|) + c}.$$

Substituting back v, u and y, we obtain $w = v(t) = u(t)/t = [y(t) + 1]/t$, so

$$\frac{y+1}{t} = -\frac{1}{\ln(|t|) + c} \quad \Leftrightarrow \quad y(t) = -\frac{t}{\ln(|t|) + c} - 1.$$

$\triangleleft$

Notes. This section corresponds to Boyce-DiPrima [3] Section 2.2. Zill and Wright study separable equations in [17] Section 2.2, and Euler homogeneous equations in Section 2.5. Zill and Wright organize the material in a nice way, they present first separable equations, then linear equations, and then they group Euler homogeneous and Bernoulli equations in a section called Solutions by Substitution. Once again, a one page description is given by Simmons in [10] in Chapter 2, Section 7.

1.3.4. Exercises.

1.3.1.- Find all solutions y to the ODE

$$y' = \frac{t^2}{y}.$$

Express the solutions in explicit form.

1.3.2.- Find every solution y of the ODE

$$3t^2 + 4y^3 y' - 1 + y' = 0.$$

Leave the solution in implicit form.

1.3.3.- Find the solution y to the IVP

$$y' = t^2 y^2, \quad y(0) = 1.$$

1.3.4.- Find every solution y of the ODE

$$ty + \sqrt{1 + t^2}\, y' = 0.$$

1.3.5.- Find every solution y of the Euler homogeneous equation

$$y' = \frac{y + t}{t}.$$

1.3.6.- Find all solutions y to the ODE

$$y' = \frac{t^2 + y^2}{ty}.$$

1.3.7.- Find the explicit solution to the IVP

$$(t^2 + 2ty)\, y' = y^2, \quad y(1) = 1.$$

1.3.8.- Prove that if $y' = f(t, y)$ is an Euler homogeneous equation and $y_1(t)$ is a solution, then $y(t) = (1/k)\, y_1(kt)$ is also a solution for every non-zero $k \in \mathbb{R}$.

1.3.9.- * Find the explicit solution of the initial value problem

$$y' = \frac{4t - 6t^2}{y}, \quad y(0) = -3.$$

1.4. Exact Differential Equations

A differential equation is exact when is a total derivative of a function, called potential function. Exact equations are simple to integrate—any potential function must be constant. The solutions of the differential equation define level surfaces of a potential function.

A semi-exact differential equation is an equation that is not exact but it can be transformed into an exact equation after multiplication by a function, called an integrating factor. An integrating factor converts a non-exact equation into an exact equation. Linear equations, studied in § 1.1 and § 1.2, are a particular case of semi-exact equations. The integrating factor of a linear equation transforms it into a total derivative—hence, an exact equation. We now generalize this idea to a class of nonlinear equations.

1.4.1. Exact Equations.

A differential equation is exact if certain parts of the differential equation have matching partial derivatives. We use this definition because it is simple to check in concrete examples.

Definition 1.4.1. *An* ***exact*** *differential equation for y is*

$$N(t,y)\,y' + M(t,y) = 0$$

where the functions N and M satisfy

$$\partial_t N(t,y) = \partial_y M(t,y)$$

Remark: The functions N, M depend on t, y, and we use the notation for partial derivatives

$$\partial_t N = \frac{\partial N}{\partial t}, \qquad \partial_y M = \frac{\partial M}{\partial y}.$$

In the definition above, the letter y has been used both as the unknown function (in the first equation), and as an independent variable (in the second equation). We use this dual meaning for the letter y throughout this section.

Our first example shows that all separable equations studied in § 1.3 are exact.

Example 1.4.1. Show whether a separable equation $h(y)\,y'(t) = g(t)$ is exact or not.

Solution: If we write the equation as $h(y)\,y' - g(t) = 0$, then

$$\left.\begin{aligned} N(t,y) = h(y) \quad &\Rightarrow \quad \partial_t N(t,y) = 0,\\ M(t,y) = g(t) \quad &\Rightarrow \quad \partial_y M(t,y) = 0, \end{aligned}\right\} \quad \Rightarrow \quad \partial_t N(t,y) = \partial_y M(t,y),$$

hence every separable equation is exact. ◁

The next example shows that linear equations, written as in § 1.2, are not exact.

Example 1.4.2. Show whether the linear differential equation below is exact or not,

$$y'(t) = a(t)\,y(t) + b(t), \qquad a(t) \neq 0.$$

Solution: We first find the functions N and M rewriting the equation as follows,

$$y' + a(t)y - b(t) = 0 \quad \Rightarrow \quad N(t,y) = 1, \quad M(t,y) = -a(t)\,y - b(t).$$

Let us check whether the equation is exact or not,

$$\left.\begin{aligned} N(t,y) = 1 \quad &\Rightarrow \quad \partial_t N(t,y) = 0,\\ M(t,y) = -a(t)y - b(t) \quad &\Rightarrow \quad \partial_y M(t,y) = -a(t), \end{aligned}\right\} \quad \Rightarrow \quad \partial_t N(t,y) \neq \partial_y M(t,y).$$

So, the differential equation is not exact. ◁

The following examples show that there are exact equations which are not separable.

Example 1.4.3. Show whether the differential equation below is exact or not,

$$2ty\, y' + 2t + y^2 = 0.$$

Solution: We first identify the functions N and M. This is simple in this case, since

$$(2ty)\, y' + (2t + y^2) = 0 \quad \Rightarrow \quad N(t,y) = 2ty, \quad M(t,y) = 2t + y^2.$$

The equation is indeed exact, since

$$\left.\begin{aligned} N(t,y) &= 2ty &\Rightarrow\quad \partial_t N(t,y) &= 2y,\\ M(t,y) &= 2t + y^2 &\Rightarrow\quad \partial_y M(t,y) &= 2y, \end{aligned}\right\} \quad \Rightarrow \quad \partial_t N(t,y) = \partial_y M(t,y).$$

Therefore, the differential equation is exact. ◁

Example 1.4.4. Show whether the differential equation below is exact or not,

$$\sin(t)\, y' + t^2 e^y\, y' - y' = -y\cos(t) - 2te^y.$$

Solution: We first identify the functions N and M by rewriting the equation as follows,

$$\big(\sin(t) + t^2 e^y - 1\big)\, y' + \big(y\cos(t) + 2te^y\big) = 0$$

we can see that

$$\begin{aligned} N(t,y) &= \sin(t) + t^2 e^y - 1 &\Rightarrow\quad \partial_t N(t,y) &= \cos(t) + 2te^y,\\ M(t,y) &= y\cos(t) + 2te^y &\Rightarrow\quad \partial_y M(t,y) &= \cos(t) + 2te^y. \end{aligned}$$

Therefore, $\partial_t N(t,y) = \partial_y M(t,y)$, and the equation is exact. ◁

1.4.2. Solving Exact Equations.

Exact differential equations can be rewritten as a total derivative of a function, called a potential function. Once they are written in such way they are simple to solve.

Theorem 1.4.2 (Exact Equations)**.** *If the differential equation*

$$N(t,y)\, y' + M(t,y) = 0 \tag{1.4.1}$$

is exact, then it can be written as

$$\frac{d\psi}{dt}(t, y(t)) = 0,$$

where ψ is called a potential function and satisfies

$$N = \partial_y \psi, \qquad M = \partial_t \psi. \tag{1.4.2}$$

Therefore, the solutions of the exact equation are given in implicit form as

$$\psi(t, y(t)) = c, \qquad c \in \mathbb{R}.$$

Remark: The condition $\partial_t N = \partial_y M$ is equivalent to the existence of a potential function—result proven by Henri Poincaré around 1880.

Theorem 1.4.3 (Poincaré)**.** *Continuously differentiable functions N, M, on t, y, satisfy*

$$\partial_t N(t,y) = \partial_y M(t,y) \tag{1.4.3}$$

iff there is a twice continuously differentiable function ψ, depending on t, y such that

$$\partial_y \psi(t,y) = N(t,y), \qquad \partial_t \psi(t,y) = M(t,y). \tag{1.4.4}$$

Remarks:

(a) A differential equation defines the functions N and M. The exact condition in (1.4.3) is equivalent to the existence of ψ, related to N and M through Eq. (1.4.4).

(b) If we recall the definition of the gradient of a function of two variables, $\nabla\psi = \langle \partial_t \psi, \partial_y \psi \rangle$, then the equations in (1.4.4) say that $\nabla\psi = \langle M, N \rangle$.

Proof of Theorem 1.4.3:

($\Rightarrow$) It is not given. See [9].

($\Leftarrow$) We assume that the potential function ψ is given and satisfies

$$N = \partial_y \psi, \qquad M = \partial_t \psi.$$

Recalling that ψ is twice continuously differentiable, hence $\partial_t \partial_y \psi = \partial_y \partial_t \psi$, then we have

$$\partial_t N = \partial_t \partial_y \psi = \partial_y \partial_t \psi = \partial_y M.$$

□

In our next example we verify that a given function ψ is a potential function for an exact differential equation. We also show that the differential equation can be rewritten as a total derivative of this potential function. (In Theorem 1.4.2 we show how to compute such potential function from the differential equation, integrating the equations in (1.4.4).)

Example 1.4.5 (Verification of a Potential)**.** Show that the differential equation

$$2ty\, y' + 2t + y^2 = 0.$$

is the total derivative of the potential function $\psi(t,y) = t^2 + ty^2$.

Solution: we use the chain rule to compute the t derivative of the potential function ψ evaluated at the unknown function y,

$$\begin{aligned} \frac{d}{dt}\psi(t,y(t)) &= \big(\partial_y \psi\big)\,\frac{dy}{dt} + \big(\partial_t \psi\big) \\ &= (2ty)\, y' + (2t + y^2). \end{aligned}$$

So the differential equation is the total derivative of the potential function. To get this result we used the partial derivatives

$$\partial_y \psi = 2ty = N, \qquad \partial_t \psi = 2t + y^2 = M.$$

◁

Exact equations always have a potential function ψ, and this function is not difficult to compute—we only need to integrate Eq. (1.4.4). Having a potential function of an exact equation is essentially the same as solving the differential equation, since the integral curves of ψ define implicit solutions of the differential equation.

Proof of Theorem 1.4.2: The differential equation in (1.4.1) is exact, then Poincaré Theorem implies that there is a potential function ψ such that

$$N = \partial_y \psi, \qquad M = \partial_t \psi.$$

Therefore, the differential equation is given by

$$\begin{aligned} 0 &= N(t,y)\,y'(t) + M(t,y) \\ &= \big(\partial_y \psi(t,y)\big)\,y' + \big(\partial_t \psi(t,y)\big) \\ &= \frac{d}{dt}\psi(t,y(t)), \end{aligned}$$

where in the last step we used the chain rule. Recall that the chain rule says

$$\frac{d}{dt}\psi\big(t,y(t)\big) = (\partial_y \psi)\,\frac{dy}{dt} + (\partial_t \psi).$$

So, the differential equation has been rewritten as a total t-derivative of the potential function, which is simple to integrate,

$$\frac{d}{dt}\psi(t,y(t)) = 0 \quad \Rightarrow \quad \psi(t,y(t)) = c,$$

where c is an arbitrary constant. This establishes the Theorem. $\square$

Example 1.4.6 (Calculation of a Potential)**.** Find all solutions y to the differential equation

$$2ty\,y' + 2t + y^2 = 0.$$

Solution: The first step is to verify whether the differential equation is exact. We know the answer, the equation is exact, we did this calculation before in Example 1.4.3, but we reproduce it here anyway.

$$\left.\begin{aligned} N(t,y) &= 2ty &\Rightarrow\quad \partial_t N(t,y) &= 2y, \\ M(t,y) &= 2t + y^2 &\Rightarrow\quad \partial_y M(t,y) &= 2y. \end{aligned}\right\} \quad\Rightarrow\quad \partial_t N(t,y) = \partial_y M(t,y).$$

Since the equation is exact, Lemma 1.4.3 implies that there exists a potential function ψ satisfying the equations

$$\partial_y \psi(t,y) = N(t,y), \tag{1.4.5}$$

$$\partial_t \psi(t,y) = M(t,y). \tag{1.4.6}$$

Let us compute ψ. Integrate Eq. (1.4.5) in the variable y keeping the variable t constant,

$$\partial_y \psi(t,y) = 2ty \quad \Rightarrow \quad \psi(t,y) = \int 2ty\,dy + g(t),$$

where g is a constant of integration on the variable y, so g can only depend on t. We obtain

$$\psi(t,y) = ty^2 + g(t). \tag{1.4.7}$$

Introduce into Eq. (1.4.6) the expression for the function ψ in Eq. (1.4.7) above, that is,

$$y^2 + g'(t) = \partial_t \psi(t,y) = M(t,y) = 2t + y^2 \quad \Rightarrow \quad g'(t) = 2t$$

Integrate in t the last equation above, and choose the integration constant to be zero,

$$g(t) = t^2.$$

We have found that a potential function is given by

$$\psi(t,y) = ty^2 + t^2.$$

Therefore, Theorem 1.4.2 implies that all solutions y satisfy the implicit equation

$$ty^2(t) + t^2 = c,$$

for any $c \in R$. The choice $g(t) = t^2 + c_0$ only modifies the constant c. $\triangleleft$

Remark: An exact equation and its solutions can be pictured on the graph of a potential function. This is called a *geometrical interpretation* of the exact equation. We saw that an exact equation $N\,y' + M = 0$ can be rewritten as $d\psi/dt = 0$. The solutions of the differential equation are functions y such that $\psi(t, y(t)) = c$, hence *the solutions define level curves of the potential function.* Given a level curve, the vector $\boldsymbol{r}(t) = \langle t, y(t)\rangle$, which belongs to the ty-plane, points to the level curve, while its derivative $\boldsymbol{r}'(t) = \langle 1, y'(t)\rangle$ is tangent to the level curve. Since the gradient vector $\nabla\psi = \langle M, N\rangle$ is perpendicular to the level curve,

$$\boldsymbol{r}' \perp \nabla\psi \quad\Leftrightarrow\quad \boldsymbol{r}'\cdot\nabla\psi = 0 \quad\Leftrightarrow\quad M + N\,y' = 0.$$

We wanted to remark that the differential equation can be thought as the condition $\boldsymbol{r}' \perp \nabla\psi$.

As an example, consider the differential equation

$$2y\,y' + 2t = 0,$$

which is separable, so it is exact. A potential function is

$$\psi = t^2 + y^2,$$

a paraboloid shown in Fig. 2. Solutions y are defined by the equation $t^2 + y^2 = c$, which are level curves of ψ for $c > 0$. The graph of a solution is shown on the ty-plane, given by

$$y(t) = \pm\sqrt{c - t^2}.$$

As we said above, the vector $\boldsymbol{r}(t) = \langle t, y(t)\rangle$ points to the solution's graph while its derivative $\boldsymbol{r}'(t) = \langle 1, y'(t)\rangle$ is tangent to the level cuve. We also know that the gradient vector $\nabla\psi = \langle 2t, 2y\rangle$ is perpendicular to the level curve. The condition

$$\boldsymbol{r}' \perp \nabla\psi \quad\Rightarrow\quad \boldsymbol{r}'\cdot\nabla\psi = 0,$$

is precisely the differential equation,

$$2t + 2y\,y' = 0.$$

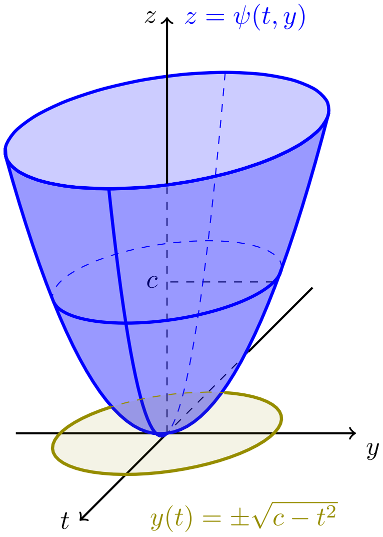


FIGURE 2. Potential ψ with level curve $\psi = c$ defines a solution y on the ty-plane.

Example 1.4.7 (Calculation of a Potential)**.** Find all solutions y to the equation

$$\sin(t)\,y' + t^2 e^y\,y' - y' + y\cos(t) + 2te^y - 3t^2 = 0.$$

Solution: The first step is to verify whether the differential equation is exact.

$$N(t,y) = \sin(t) + t^2 e^y - 1 \quad\Rightarrow\quad \partial_t N(t,y) = \cos(t) + 2te^y,$$
$$M(t,y) = y\cos(t) + 2te^y - 3t^2 \quad\Rightarrow\quad \partial_y M(t,y) = \cos(t) + 2te^y.$$

So, the equation is exact. Poincaré Theorem says there is a potential function ψ satisfying

$$\partial_y\psi(t,y) = N(t,y), \qquad \partial_t\psi(t,y) = M(t,y). \tag{1.4.8}$$

To compute ψ we integrate on y the equation $\partial_y\psi = N$ keeping t constant,

$$\partial_y\psi(t,y) = \sin(t) + t^2 e^y - 1 \quad\Rightarrow\quad \psi(t,y) = \int \big(\sin(t) + t^2 e^y - 1\big)\,dy + g(t)$$

where g is a constant of integration on the variable y, so g can only depend on t. We obtain

$$\psi(t,y) = y\sin(t) + t^2 e^y - y + g(t).$$

Now introduce the expression above for ψ in the second equation in Eq. (1.4.8),

$$y\cos(t) + 2te^y + g'(t) = \partial_t\psi(t,y) = M(t,y) = y\cos(t) + 2te^y - 3t^2 \quad \Rightarrow \quad g'(t) = -3t^2.$$

The solution is $g(t) = -t^3 + c_0$, with c_0 a constant. We choose $c_0 = 0$, so $g(t) = -t^3$. We found g, so we have the complete potential function,

$$\psi(t,y) = y\sin(t) + t^2e^y - y - t^3.$$

Theorem 1.4.2 implies that any solution y satisfies the implicit equation

$$y(t)\sin(t) + t^2e^{y(t)} - y(t) - t^3 = c.$$

The solution y above cannot be written in explicit form. If we choose the constant $c_0 \neq 0$ in $g(t) = -t^3 + c_0$, we only modify the constant c above. ◁

Remark: A potential function is also called a *conserved quantity*. This is a reasonable name, since a potential function evaluated at any solution of the differential equation is constant along the evolution. This is yet another interpretation of the equation $d\psi/dt = 0$, or its integral $\psi(t,y(t)) = c$. If we call $c = \psi_0 = \psi(0,y(0))$, the value of the potential function at the initial conditions, then $\psi(t,y(t)) = \psi_0$.

Conserved quantities are important in physics. The energy of a moving particle is a famous conserved quantity. In that case the differential equation is Newton's second law of motion, mass times acceleration equals force. One can prove that the energy E of a particle with position function y moving under a conservative force is kept constant in time. This statement can be expressed by $E(t,y(t),y'(t)) = E_0$, where E_0 is the particle's energy at the initial time.

1.4.3. Semi-Exact Equations.

Sometimes a non-exact differential equation can be rewritten as an exact equation. One way this could happen is multiplying the differential equation by an appropriate function. If the new equation is exact, the multiplicative function is called an integrating factor.

Definition 1.4.4. *A* ***semi-exact*** *differential equation is a non-exact equation that can be transformed into an exact equation after a multiplication by an integrating factor.*

Example 1.4.8. Show that linear differential equations $y' = a(t)\,y + b(t)$ are semi-exact.

Solution: We first show that linear equations $y' = a\,y + b$ with $a \neq 0$ are not exact. If we write them as

$$y' - a\,y - b = 0 \quad \Rightarrow \quad N\,y' + M = 0 \quad \text{with} \quad N = 1, \quad M = -a\,y - b.$$

Therefore,

$$\partial_t N = 0, \quad \partial_y M = -a \quad \Rightarrow \quad \partial_t N \neq \partial_y M.$$

We now show that linear equations are semi-exact. Let us multiply the linear equation by a function μ, which depends only on t,

$$\mu(t)\,y' - a(t)\,\mu(t)\,y - \mu(t)\,b(t) = 0,$$

where we emphasized that μ, a, b depende only on t. Let us look for a particular function μ that makes the equation above exact. If we write this equation as $\tilde{N}\,y' + \tilde{M} = 0$, then

$$\tilde{N}(t,y) = \mu, \quad \tilde{M}(t,y) = -a\,\mu\,y - \mu\,b.$$

We now check the condition for exactness,

$$\partial_t \tilde{N} = \mu', \quad \partial_y \tilde{M} = -a\,\mu,$$

and we get that

$$\left.\begin{array}{c} \partial_t \tilde{N} = \partial_y \tilde{M} \\ \text{the equation is exact} \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{c} \mu' = -a\,\mu \\ \mu \text{ is an integrating factor.} \end{array}\right.$$

Therefore, the linear equation $y' = a\,y + b$ is semi-exact, and the function that transforms it into an exact equation is $\mu(t) = e^{-A(t)}$, where $A(t) = \int a(t)\,dt$, which in § 1.2 we called it an integrating factor. ◁

Now we generalize this idea to nonlinear differential equations.

Theorem 1.4.5. *If the equation*

$$N(t,y)\,y' + M(t,y) = 0 \tag{1.4.9}$$

is not exact, with $\partial_t N \neq \partial_y M$, the function $N \neq 0$, and where the function h defined as

$$h = \frac{\partial_y M - \partial_t N}{N} \tag{1.4.10}$$

depends only on t, not on y, then the equation below is exact,

$$(e^H N)\,y' + (e^H M) = 0, \tag{1.4.11}$$

where H is an antiderivative of h,

$$H(t) = \int h(t)\,dt.$$

Remarks:

(a) The function $\mu(t) = e^{H(t)}$ is called an *integrating factor*.

(b) Any integrating factor μ is solution of the differential equation

$$\mu'(t) = h(t)\,\mu(t).$$

(c) Multiplication by an integrating factor transforms a non-exact equation

$$N\,y' + M = 0$$

into an exact equation.

$$(\mu N)\,y' + (\mu M) = 0.$$

This is exactly what happened with linear equations.

Verification Proof of Theorem 1.4.5: We need to verify that the equation is exact,

$$(e^H\,N)\,y' + (e^H\,M) = 0 \quad \Rightarrow \quad \tilde{N}(t,y) = e^{H(t)}\,N(t,y), \quad \tilde{M}(t,y) = e^{H(t)}\,M(t,y).$$

We now check for exactness, and let us recall $\partial_t(e^H) = (e^H)' = h\,e^H$, then

$$\partial_t \tilde{N} = h\,e^H\,N + e^H\,\partial_t N, \qquad \partial_y \tilde{M} = e^H\,\partial_y M.$$

Let us use the definition of h in the first equation above,

$$\partial_t \tilde{N} = e^H \Big(\frac{(\partial_y M - \partial_t N)}{N}\,N + \partial_t N \Big) = e^H\,\partial_y M = \partial_y \tilde{M}.$$

So the equation is exact. This establishes the Theorem. □

Constructive Proof of Theorem 1.4.5: The original differential equation

$$N\,y' + M = 0$$

is not exact because $\partial_t N \neq \partial_y M$. Now multiply the differential equation by a nonzero function μ that depends only on t,

$$(\mu N)\,y' + (\mu M) = 0. \tag{1.4.12}$$

We look for a function μ such that this new equation is exact. This means that μ must satisfy the equation

$$\partial_t(\mu N) = \partial_y(\mu M).$$

Recalling that μ depends only on t and denoting $\partial_t \mu = \mu'$, we get

$$\mu'\,N + \mu\,\partial_t N = \mu\,\partial_y M \quad \Rightarrow \quad \mu'\,N = \mu\,(\partial_y M - \partial_t N).$$

So the differential equation in (1.4.12) is exact iff holds

$$\mu' = \Big(\frac{\partial_y M - \partial_t N}{N}\Big)\,\mu.$$

The solution μ will depend only on t iff the function

$$h(t) = \frac{\partial_y M(t,y) - \partial_t N(t,y)}{N(t,y)}$$

depends only on t. If this happens, as assumed in the hypotheses of the theorem, then we can solve for μ as follows,

$$\mu'(t) = h(t)\,\mu(t) \quad \Rightarrow \quad \mu(t) = e^{H(t)}, \quad H(t) = \int h(t)\,dt.$$

Therefore, the equation below is exact,

$$(e^H\,N)\,y' + (e^H\,M) = 0.$$

This establishes the Theorem. □

Example 1.4.9. Find all solutions y to the differential equation

$$\big(t^2 + t\,y\big)\,y' + \big(3t\,y + y^2\big) = 0. \tag{1.4.13}$$

Solution: We first verify whether this equation is exact:

$$\begin{aligned} N(t,y) &= t^2 + ty \qquad &\Rightarrow& \qquad &\partial_t N(t,y) = 2t + y,\\ M(t,y) &= 3ty + y^2 \qquad &\Rightarrow& \qquad &\partial_y M(t,y) = 3t + 2y, \end{aligned}$$

therefore, the differential equation is not exact. We now verify whether the extra condition in Theorem 1.4.5 holds, that is, whether the function in (1.4.10) is y independent;

$$\begin{aligned} h &= \frac{\partial_y M(t,y) - \partial_t N(t,y)}{N(t,y)}\\ &= \frac{(3t+2y) - (2t+y)}{(t^2+ty)}\\ &= \frac{(t+y)}{t(t+y)}\\ &= \frac{1}{t} \quad \Rightarrow \quad h(t) = \frac{1}{t}. \end{aligned}$$

So, the function $h = (\partial_y M - \partial_t N)/N$ is y independent. Therefore, Theorem 1.4.5 implies that the non-exact differential equation can be transformed into an exact equation. We need to multiply the differential equation by a function μ solution of the equation

$$\mu'(t) = h(t)\,\mu(t) \quad \Rightarrow \quad \frac{\mu'}{\mu} = \frac{1}{t} \quad \Rightarrow \quad \ln(\mu(t)) = \ln(t) \quad \Rightarrow \quad \mu(t) = t,$$

where we have chosen in second equation the integration constant to be zero. Then, multiplying the original differential equation in (1.4.13) by the integrating factor μ we obtain

$$\big(3t^2\,y + t\,y^2\big) + \big(t^3 + t^2\,y\big)\,y' = 0. \tag{1.4.14}$$

This latter equation is exact, since

$$\begin{aligned} \tilde{N}(t,y) &= t^3 + t^2 y \qquad &\Rightarrow \qquad \partial_t \tilde{N}(t,y) &= 3t^2 + 2ty,\\ \tilde{M}(t,y) &= 3t^2 y + t y^2 \qquad &\Rightarrow \qquad \partial_y \tilde{M}(t,y) &= 3t^2 + 2ty, \end{aligned}$$

so we get the exactness condition $\partial_t \tilde{N} = \partial_y \tilde{M}$. The solution y can be found as we did in the previous examples in this Section. That is, we find the potential function ψ by integrating the equations

$$\partial_y \psi(t,y) = \tilde{N}(t,y), \tag{1.4.15}$$

$$\partial_t \psi(t,y) = \tilde{M}(t,y). \tag{1.4.16}$$

From the first equation above we obtain

$$\partial_y \psi = t^3 + t^2 y \quad \Rightarrow \quad \psi(t,y) = \int \big(t^3 + t^2 y\big)\,dy + g(t).$$

Integrating on the right hand side above we arrive to

$$\psi(t,y) = t^3 y + \frac{1}{2}\,t^2 y^2 + g(t).$$

Introduce the expression above for ψ in Eq. (1.4.16),

$$3t^2 y + t y^2 + g'(t) = \partial_t \psi(t,y) = \tilde{M}(t,y) = 3t^2 y + t y^2,$$
$$g'(t) = 0.$$

A solution to this last equation is $g(t) = 0$. So we get a potential function

$$\psi(t,y) = t^3 + \frac{1}{2}\,t^2 y^2.$$

All solutions y to the differential equation in (1.4.13) satisfy the equation

$$t^3\,y(t) + \frac{1}{2}\,t^2\big(y(t)\big)^2 = c_0,$$

where $c_0 \in \mathbb{R}$ is arbitrary. ◁

We have seen in Example 1.4.2 that linear differential equations with $a \neq 0$ are not exact. In Section 1.2 we found solutions to linear equations using the integrating factor method. We multiplied the linear equation by a function that transformed the equation into a total derivative. Those calculations are now a particular case of Theorem 1.4.5, as we can see it in the following Example.

Example 1.4.10. Use Theorem 1.4.5 to find all solutions to the linear differential equation

$$y' = a(t)\,y + b(t), \qquad a(t) \neq 0. \tag{1.4.17}$$

Solution: We first write the linear equation in a way we can identify functions N and M,

$$y' - \big(a(t)\, y + b(t)\big) = 0.$$

We now verify whether the linear equation is exact or not. Actually, we have seen in Example 1.4.3 that this equation is not exact, since

$$\begin{aligned} N(t,y) &= 1 & &\Rightarrow & \partial_t N(t,y) &= 0, \\ M(t,y) &= -a(t)\, y - b(t) & &\Rightarrow & \partial_y M(t,y) &= -a(t). \end{aligned}$$

But now we can go further, we can check wheteher the condtion in Theorem 1.4.5 holds or not. We compute the function

$$\frac{\partial_y M(t,y) - \partial_t N(t,y)}{N(t,y)} = \frac{-a(t) - 0}{1} = -a(t)$$

and we see that it is independent of the variable y. Theorem 1.4.5 says that we can transform the linear equation into an exact equation. We only need to multiply the linear equation by a function μ, solution of the equation

$$\mu'(t) = -a(t)\, \mu(t) \quad \Rightarrow \quad \mu(t) = e^{-A(t)}, \quad A(t) = \int a(t)\, dt.$$

This is the same integrating factor we discovered in Section 1.2. Therefore, the equation below is exact,

$$e^{-A(t)}\, y' - \big(a(t)\, e^{-A(t)}\, y - b(t)\, e^{-A(t)}\big) = 0. \tag{1.4.18}$$

This new version of the linear equation is exact, since

$$\begin{aligned} \tilde{N}(t,y) &= e^{-A(t)} & &\Rightarrow & \partial_t \tilde{N}(t,y) &= -a(t)\, e^{-A(t)}, \\ \tilde{M}(t,y) &= -a(t)\, e^{-A(t)}\, y - b(t)\, e^{-A(t)} & &\Rightarrow & \partial_y \tilde{M}(t,y) &= -a(t)\, e^{-A(t)}. \end{aligned}$$

Since the linear equation is now exact, the solutions y can be found as we did in the previous examples in this Section. We find the potential function ψ integrating the equations

$$\partial_y \psi(t,y) = \tilde{N}(t,y), \tag{1.4.19}$$

$$\partial_t \psi(t,y) = \tilde{M}(t,y). \tag{1.4.20}$$

From the first equation above we obtain

$$\partial_y \psi = e^{-A(t)} \quad \Rightarrow \quad \psi(t,y) = \int e^{-A(t)}\, dy + g(t).$$

The integral is simple, since $e^{-A(t)}$ is y independent. We then get

$$\psi(t,y) = e^{-A(t)}\, y + g(t).$$

We introduce the expression above for ψ in Eq. (1.4.16),

$$-a(t)\, e^{-A(t)}\, y + g'(t) = \partial_t \psi(t,y) = \tilde{M}(t,y) = -a(t)\, e^{-A(t)}\, y - b(t)\, e^{-A(t)},$$

$$g'(t) = -b(t)\, e^{-A(t)}.$$

A solution for function g is then given by

$$g(t) = -\int b(t)\, e^{-A(t)}\, dt.$$

Having that function g, we get a potential function

$$\psi(t,y) = e^{-A(t)}\, y - \int b(t)\, e^{-A(t)}\, dt.$$

All solutions y to the linear differential equation in (1.4.17) satisfy the equation

$$e^{-A(t)}\,y(t) - \int b(t)\,e^{-A(t)}\,dt = c_0,$$

where $c_0 \in \mathbb{R}$ is arbitrary. This is the implicit form of the solution, but in this case it is simple to find the explicit form too,

$$y(t) = e^{A(t)}\Big(c_0 + \int b(t)\,e^{-A(t)}\,dt\Big).$$

This expression agrees with the one in Theorem 1.2.3, when we studied linear equations. ◁

1.4.4. The Equation for the Inverse Function.

Sometimes the equation for a function y is neither exact nor semi-exact, but the equation for the inverse function y^{-1} might be. We now try to find out when this can happen. To carry out this study it is more convenient to change a little bit the notation we have been using so far:

(a) We change the independent variable name from t to x. Therefore, we write differential equations as

$$N(x,y)\,y' + M(x,y) = 0, \qquad y = y(x), \qquad y' = \frac{dy}{dx}.$$

(b) We denote by $x(y)$ the inverse of $y(x)$, that is,

$$x(y_1) = x_1 \quad\Leftrightarrow\quad y(x_1) = y_1.$$

(c) Recall the identity relating derivatives of a function and its inverse function,

$$x'(y) = \frac{1}{y'(x)}.$$

Our first result says that for exact equations it makes no difference to solve for y or its inverse x. If one equation is exact, so is the other equation.

Theorem 1.4.6. *$N\,y' + M = 0$ is exact $\quad\Leftrightarrow\quad$ $M\,x' + N = 0$ is exact.*

Remark: We will see that for semi-exact equations there is a difference.

Proof of Theorem 1.4.6: Write the differential equation of a function y with values $y(x)$,

$$N(x,y)\,y' + M(x,y) = 0 \quad\text{and}\quad \partial_x N = \partial_y M.$$

If a solution y is invertible we denote $y^{-1}(y) = x(y)$, and we have the well-known relation

$$x'(y) = \frac{1}{y'(x(y))}.$$

Divide the differential equation above by y' and use the relation above, then we get

$$N(x,y) + M(x,y)\,x' = 0,$$

where now y is the independent variable and the unknwon function is x, with values $x(y)$, and the prime means $x' = dx/dy$. The condition for this last equation to be exact is

$$\partial_y M = \partial_x N,$$

which is exactly the same condition for the equation $N\,y' + M = 0$ to be exact. This establishes the Theorem. □

Remark: Sometimes, in the literature, the equations $N\,y' + M = 0$ and $N + M\,x' = 0$ are written together as follows,

$$N\,dy + M\,dx = 0.$$

This equation deserves two comments:

(a) We do not use this notation here. That equation makes sense in the framework of differential forms, which is beyond the subject of these notes.

(b) Some people justify the use of that equation outside the framework of differential forms by thinking $y' = \dfrac{dy}{dx}$ as real fraction and multiplying $N\,y' + M = 0$ by the denominator,

$$N\,\frac{dy}{dx} + M = 0 \quad \Rightarrow \quad N\,dy + M\,dx = 0.$$

Unfortunately, y' is not a fraction $\dfrac{dy}{dx}$, so the calculation just mentioned has no meaning.

So, if the equation for y is exact, so is the equation for its inverse x. The same is not true for semi-exact equations. If the equation for y is semi-exact, then the equation for its inverse x might or might not be semi-exact. The next result states a condition on the equation for the inverse function x to be semi-exact. This condition is not equal to the condition on the equation for the function y to be semi-exact. Compare Theorems 1.4.5 and 1.4.7.

Theorem 1.4.7. *If the equation*

$$M\,x' + N = 0$$

is not exact, with $\partial_y M \neq \partial_x N$, the function $M \neq 0$, and where the function ℓ defined as

$$\ell = -\frac{(\partial_y M - \partial_x N)}{M}$$

depends only on y, not on x, then the equation below is exact,

$$(e^L M)\,x' + (e^L N) = 0$$

where L is an antiderivative of ℓ,

$$L(y) = \int \ell(y)\,dy.$$

Remarks:

(a) The function $\mu(y) = e^{L(y)}$ is called an *integrating factor*.

(b) Any integrating factor μ is solution of the differential equation

$$\mu'(y) = \ell(y)\,\mu(y).$$

(c) Multiplication by an integrating factor transforms a non-exact equation

$$M\,x' + N = 0$$

into an exact equation.

$$(\mu M)\,x' + (\mu N) = 0.$$

Verification Proof of Theorem 1.4.7: We need to verify that the equation is exact,

$$(e^L\,M)\,x' + (e^L\,N) = 0 \quad \Rightarrow \quad \tilde M(x,y) = e^{L(y)}\,M(x,y), \quad \tilde N(x,y) = e^{L(y)}\,N(x,y).$$

We now check for exactness, and let us recall $\partial_y(e^L) = (e^L)' = \ell\,e^L$, then

$$\partial_y \tilde M = \ell\,e^L\,M + e^L\,\partial_y M, \qquad \partial_x \tilde N = e^H\,\partial_x N.$$

Let us use the definition of ℓ in the first equation above,

$$\partial_y \tilde{M} = e^L \Big(-\frac{(\partial_y M - \partial_x N)}{M}\, M + \partial_y M \Big) = e^L\, \partial_x N = \partial_x \tilde{N}.$$

So the equation is exact. This establishes the Theorem. $\square$

Constructive Proof of Theorem 1.4.7: The original differential equation

$$M\, x' + N = 0$$

is not exact because $\partial_y M \neq \partial_x N$. Now multiply the differential equation by a nonzero function μ that depends only on y,

$$(\mu M)\, x' + (\mu N) = 0.$$

We look for a function μ such that this new equation is exact. This means that μ must satisfy the equation

$$\partial_y(\mu M) = \partial_x(\mu N).$$

Recalling that μ depends only on y and denoting $\partial_y \mu = \mu'$, we get

$$\mu'\, M + \mu\, \partial_y M = \mu\, \partial_x N \quad \Rightarrow \quad \mu'\, M = -\mu\, (\partial_y M - \partial_x N).$$

So the differential equation $(\mu M)\, x' + (\mu N) = 0$ is exact iff holds

$$\mu' = -\Big(\frac{\partial_y M - \partial_x N}{M} \Big)\, \mu.$$

The solution μ will depend only on y iff the function

$$\ell(y) = -\frac{\partial_y M(x,y) - \partial_x N(x,y)}{M(x,y)}$$

depends only on y. If this happens, as assumed in the hypotheses of the theorem, then we can solve for μ as follows,

$$\mu'(y) = \ell(y)\, \mu(y) \quad \Rightarrow \quad \mu(y) = e^{L(y)}, \quad L(y) = \int \ell(y)\, dy.$$

Therefore, the equation below is exact,

$$(e^L\, M)\, x' + (e^L\, N) = 0.$$

This establishes the Theorem. $\square$

Example 1.4.11. Find all solutions to the differential equation

$$\big(5x\, e^{-y} + 2\cos(3x)\big)\, y' + \big(5\, e^{-y} - 3\sin(3x)\big) = 0.$$

Solution: We first check if the equation is exact for the unknown function y, which depends on the variable x. If we write the equation as $N\, y' + M = 0$, with $y' = dy/dx$, then

$$N(x,y) = 5x\, e^{-y} + 2\cos(3x) \quad \Rightarrow \quad \partial_x N(x,y) = 5\, e^{-y} - 6\sin(3x),$$
$$M(x,y) = 5\, e^{-y} - 3\sin(3x) \quad \Rightarrow \quad \partial_y M(x,y) = -5\, e^{-y}.$$

Since $\partial_x N \neq \partial_y M$, the equation is not exact. Let us check if there exists an integrating factor μ that depends only on x. Following Theorem 1.4.5 we study the function

$$h = \frac{\big(\partial_y M - \partial_x N\big)}{N} = \frac{-10\, e^{-y} + 6\sin(3x)}{5x\, e^{-y} + 2\cos(3x)},$$

which is a function of both x and y and cannot be simplified into a function of x alone. Hence an integrating factor cannot be function of only x.

Let us now consider the equation for the inverse function x, which depends on the variable y. The equation is $M\,x' + N = 0$, with $x' = dx/dy$, where M and N are the same as before,

$$M(x,y) = 5\,e^{-y} - 3\sin(3x) \qquad N(x,y) = 5x\,e^{-y} + 2\cos(3x).$$

We know from Theorem 1.4.6 that this equation is not exact. Both the equation for y and equation for its inverse x must satisfy the same condition to be exact. The condition is $\partial_x N = \partial_y M$, but we have seen that this is not true for the equation in this example. The last thing we can do is to check if the equation for the inverse function x has an integrating factor μ that depends only on y. Following Theorem 1.4.7 we study the function

$$\ell = -\frac{(\partial_y M - \partial_x N)}{M} = -\frac{\big(-10\,e^{-y} + 6\sin(3x)\big)}{\big(5\,e^{-y} - 3\sin(3x)\big)} = 2 \quad \Rightarrow \quad \ell(y) = 2.$$

The function above does not depend on x, so we can solve the differential equation for $\mu(y)$,

$$\mu'(y) = \ell(y)\,\mu(y) \quad \Rightarrow \quad \mu'(y) = 2\,\mu(y) \quad \Rightarrow \quad \mu(y) = \mu_0\,e^{2y}.$$

Since μ is an integrating factor, we can choose $\mu_0 = 1$, hence $\mu(y) = e^{2y}$. If we multiply the equation for x by this integrating factor we get

$$e^{2y}\,\big(5\,e^{-y} - 3\sin(3x)\big)\,x' + e^{2y}\,\big(5x\,e^{-y} + 2\cos(3x)\big) = 0,$$

$$\big(5\,e^{y} - 3\sin(3x)\,e^{2y}\big)\,x' + \big(5x\,e^{y} + 2\cos(3x)\,e^{2y}\big) = 0.$$

This equation is exact, because if we write it as $\tilde M\,x' + \tilde N = 0$, then

$$\tilde M(x,y) = 5\,e^{y} - 3\sin(3x)\,e^{2y} \quad \Rightarrow \quad \partial_y \tilde M(x,y) = 5\,e^{y} - 6\sin(3x)\,e^{2y},$$

$$\tilde N(x,y) = 5x\,e^{y} + 2\cos(3x)\,e^{2y} \quad \Rightarrow \quad \partial_x N(x,y) = 5\,e^{y} - 6\sin(3x)\,e^{2y},$$

that is $\partial_y \tilde M = \partial_x \tilde N$. Since the equation is exact, we find a potential function ψ from

$$\partial_x \psi = \tilde M, \qquad \partial_y \psi = \tilde N.$$

Integrating on the variable x the equation $\partial_x \psi = \tilde M$ we get

$$\psi(x,y) = 5x\,e^{y} + \cos(3x)\,e^{2y} + g(y).$$

Introducing this expression for ψ into the equation $\partial_y \psi = \tilde N$ we get

$$5x\,e^{y} + 2\cos(3x)\,e^{2y} + g'(y) = \partial_y \psi = \tilde N = 5x\,e^{y} + 2\cos(3x)\,e^{2y},$$

hence $g'(y) = 0$, so we choose $g = 0$. A potential function for the equation for x is

$$\psi(x,y) = 5x\,e^{y} + \cos(3x)\,e^{2y}.$$

The solutions x of the differential equation are given by

$$5\,x(y)\,e^{y} + \cos(3\,x(y))\,e^{2y} = c.$$

Once we have the solution for the inverse function x we can find the solution for the original unknown y, which are given by

$$5x\,e^{y(x)} + \cos(3x)\,e^{2\,y(x)} = c$$

$\lhd$

Notes. Exact differential equations are studied in Boyce-DiPrima [3], Section 2.6, and in most differential equation textbooks.

1.4.5. Exercises.

1.4.1.- Consider the equation

$$(1+t^2)\,y' = -2t\,y.$$

(a) Determine whether the differential equation is exact.
(b) Find every solution of the equation above.

1.4.2.- Consider the equation

$$t\cos(y)\,y' - 2y\,y' = -t - \sin(y).$$

(a) Determine whether the differential equation is exact.
(b) Find every solution of the equation above.

1.4.3.- Consider the equation

$$y' = \frac{-2 - y\,e^{ty}}{-2y + t\,e^{ty}}.$$

(a) Determine whether the differential equation is exact.
(b) Find every solution of the equation above.

1.4.4.- Consider the equation

$$(6x^5 - xy) + (-x^2 + xy^2)y' = 0,$$

with initial condition $y(0) = 1$.

(a) Find an integrating factor μ that converts the equation above into an exact equation.
(b) Find an implicit expression for the solution y of the IVP.

1.4.5.- Consider the equation

$$\Big(2x^2y + \frac{y}{x^2}\Big)\,y' + 4xy^2 = 0,$$

with initial condition $y(0) = -2$.

(a) Find an integrating factor μ that converts the equation above into an exact equation.
(b) Find an implicit expression for the solution y of the IVP.
(c) Find the explicit expression for the solution y of the IVP.

1.4.6.- Consider the equation

$$\big(-3x\,e^{-2y} + \sin(5x)\big)\,y'$$
$$+\ \big(3\,e^{-2y} + 5\cos(5x)\big) = 0.$$

(a) Is this equation for y exact? If not, does this equation have an integrating factor depending on x?
(b) Is the equation for $x = y^{-1}$ exact? If not, does this equation have an integrating factor depending on y?
(c) Find an implicit expression for all solutions y of the differential equation above.

1.4.7.- * Find the solution to the equation

$$2t^2y + 2t^2y^2 + 1 + (t^3 + 2t^3y + 2ty)\,y' = 0,$$

with initial condition

$$y(1) = 2.$$

1.5. Applications of Linear Equations

Different physical systems may be described by the same mathematical structure. The radioactive decay of a substance, the cooling of a material, or the salt concentration on a water tank can be described with linear differential equations. A radioactive substance decays at a rate proportional to the substance amount at the time. Something similar happens to the temperature of a cooling body. Linear, constant coefficients, differential equations describe these two situations. The salt concentration inside a water tank changes in the case that salty water is allowed in and out of the tank. This situation is described with a linear variable coefficients differential equation.

1.5.1. Exponential Decay.

An example of exponential decay is the radioactive decay of certain substances, such as Uranium-235, Radium-226, Radon-222, Polonium-218, Lead-214, Cobalt-60, Carbon-14, etc. These nuclei break into several smaller nuclei and radiation. The radioactive decay of a single nucleus cannot be predicted, but the decay of a large number can. The rate of change in the amount of a radioactive substance in a sample is proportional to the negative of that amount.

Definition 1.5.1. *The **exponential decay** equation for N with decay constant $k > 0$ is*

$$N' = -k\,N.$$

Remark: The equation $N' = k\,N$, with $k > 0$ is called the *exponential growth* equation.

We have seen in § 1.1 how to solve this equation. But we review it here one more time.

Theorem 1.5.2 (Exponential Decay)**.** *The solution N of the exponential decay equation $N' = -k\,N$ and intial condition $N(0) = N_0$ is*

$$N(t) = N_0\, e^{-kt}.$$

Proof of Theorem 1.5.2: The differential equation above is both linear and separable. We choose to solve it using the integrating factor method. The integrating factor is e^{kt},

$$\big(N' + k\,N\big)e^{kt} = 0 \quad \Rightarrow \quad \big(e^{kt}N\big)' = 0 \quad \Rightarrow \quad e^{kt}N = c, \quad c \in \mathbb{R}.$$

The initial condition $N_0 = N(0) = c$, so the solution of the initial value problem is

$$N(t) = N_0\, e^{-kt}.$$

This establishes the Theorem. □

Remark: Radioactive materials are often characterized not by their decay constant k but by their half-life τ. This is a time it takes for half the material to decay.

Definition 1.5.3. *The **half-life** of a radioactive substance is the time τ such that*

$$N(\tau) = \frac{N(0)}{2}.$$

There is a simple relation between the material constant and the material half-life.

Theorem 1.5.4. *A radioactive material constant k and half-life τ are related by the equation*

$$k\tau = \ln(2).$$

Proof of Theorem 1.5.4: We know that the amount of a radioactive material as function of time is given by

$$N(t) = N_0\, e^{-kt}.$$

Then, the definition of half-life implies,

$$\frac{N_0}{2} = N_0\, e^{-k\tau} \quad \Rightarrow \quad -k\tau = \ln\Big(\frac{1}{2}\Big) \quad \Rightarrow \quad k\tau = \ln(2).$$

This establishes the Theorem. $\square$

Remark: A radioactive material, N, can be expressed in terms of the half-life,

$$N(t) = N_0\, e^{(-t/\tau)\ln(2)} \quad \Rightarrow \quad N(t) = N_0\, e^{\ln[2^{(-t/\tau)}]} \quad \Rightarrow \quad N(t) = N_0\, 2^{-t/\tau}.$$

From this last expression is clear that for $t = \tau$ we get $N(\tau) = N_0/2$.

1.5.2. Carbon-14 Dating.

Carbon-14 is a radioactive isotope of Carbon-12. An atom is an *isotope* of another atom if their nuclei have the same number of protons but different number of neutrons. The Carbon atom has 6 protons. The stable Carbon atom has also 6 neutrons, so it is called Carbon-12. Carbon-13 is another stable isotope of Carbon having 7 neutrons. Carbon-14 has 8 neutrons and it happens to be radioactive with half-life $\tau = 5730$ years. The Carbon on Earth is made up of 99% of Carbon-12 and almost 1% of Carbon-13. Carbon-14 is very rare, in the atmosphere there is 1 Carbon-14 atom per 10^{12} Carbon-12 atoms.

Carbon-14 is being constantly created in the upper atmosphere—by collisions of Carbon-12 with outer space radiation—in such a way that the proportion of Carbon-14 and Carbon-12 in the atmosphere is constant in time. The Carbon atoms are accumulated by living organisms in that same proportion. When the organism dies, the amount of Carbon-14 in the dead body decays while the amount of Carbon-12 remains constant. The proportion between radioactive over normal Carbon isotopes in the dead body decays in time. Therefore, one can measure this proportion in old remains and then find out how old are such remains—this is called Carbon-14 dating.

Example 1.5.1. Bone remains in an ancient excavation site contain only 14% of the Carbon-14 found in living animals today. Estimate how old are the bone remains. Use that the half-life of the Carbon-14 is $\tau = 5730$ years.

Solution: Suppose that $t = 0$ is set at the time when the organism dies. If at the present time t_1 the remains contain 14% of the original amount, that means

$$N(t_1) = \frac{14}{100}\, N(0).$$

Since Carbon-14 is a radioactive substance with half-life τ, the amount of Carbon-14 decays in time as follows,

$$N(t) = N(0)\, 2^{-t/\tau},$$

where $\tau = 5730$ years is the Carbon-14 half-life. Therefore,

$$2^{-t_1/\tau} = \frac{14}{100} \quad \Rightarrow \quad -\frac{t_1}{\tau} = \log_2(14/100) \quad \Rightarrow \quad t_1 = \tau \log_2(100/14).$$

We obtain that $t_1 = 16,253$ years. The organism died more that $16,000$ years ago. $\lhd$

Solution: (Using the decay constant k.) We write the solution of the radioactive decay equation as

$$N(t) = N(0)\, e^{-kt}, \qquad k\tau = \ln(2).$$

Write the condition for t_1, to be 14 % of the original Carbon-14, as follows,

$$N(0)\, e^{-kt_1} = \frac{14}{100}\, N(0) \quad \Rightarrow \quad e^{-kt_1} = \frac{14}{100} \quad \Rightarrow \quad -kt_1 = \ln\Big(\frac{14}{100}\Big),$$

so, $t_1 = \dfrac{1}{k}\, \ln\Big(\dfrac{100}{14}\Big)$. Recalling the expression for k in terms of τ, that is $k\tau = \ln(2)$, we get

$$t_1 = \tau\, \frac{\ln(100/14)}{\ln(2)}.$$

We get $t_1 = 16,253$ years, which is the same result as above, since

$$\log_2(100/14) = \frac{\ln(100/14)}{\ln(2)}.$$

◁

1.5.3. Newton's Cooling Law. In 1701 Newton published, anonymously, the result of his home made experiments done fifteen years earlier. He focused on the time evolution of the temperature of objects that rest in a medium with constant temperature. He found that the difference between the temperatues of an object and the constant temperature of a medium varies geometrically towards zero as time varies arithmetically. This was his way of saying that the difference of temperatures, ΔT, depends on time as

$$(\Delta T)(t) = (\Delta T)_0\, e^{-t/\tau},$$

for some initial temperature difference $(\Delta T)_0$ and some time scale τ. Although this is called a "Cooling Law", it also describes objects that warm up. When $(\Delta T)_0 > 0$, the object is cooling down, but when $(\Delta T)_0 < 0$, the object is warming up.

Newton knew pretty well that the function ΔT above is solution of a very particular differential equation. But he chose to put more emphasis in the solution rather than in the equation. Nowadays people think that differential equations are more fundamental than their solutions, so we define Newton's cooling law as follows.

Definition 1.5.5. *The* ***Newton cooling law*** *says that the temperature T at a time t of a material placed in a surrounding medium kept at a constant temperature T_s satisfies*

$$(\Delta T)' = -k\,(\Delta T),$$

with $\Delta T(t) = T(t) - T_s$, and $k > 0$, constant, characterizing the material thermal properties.

Remark: Newton's cooling law for ΔT is the same as the radioactive decay equation. But now the initial temperature difference, $(\Delta T)(0) = T(0) - T_s$, can be either positive or negative.

Theorem 1.5.6. *The solution of Newton's cooling law equation $(\Delta T)' = -k\,(\Delta T)$ with initial data $T(0) = T_0$ is*

$$T(t) = (T_0 - T_s)\, e^{-kt} + T_s.$$

Proof of Theorem 1.5.6: Newton's cooling law is a first order linear equation, which we solved in § 1.1. The general solution is

$$(\Delta T)(t) = c\, e^{-kt} \quad \Rightarrow \quad T(t) = c\, e^{-kt} + T_s, \qquad c \in \mathbb{R},$$

where we used that $(\Delta T)(t) = T(t) - T_s$. The initial condition implies

$$T_0 = T(0) = c + T_s \quad \Rightarrow \quad c = T_0 - T_s \quad \Rightarrow \quad T(t) = (T_0 - T_s)\, e^{-kt} + T_s.$$

This establishes the Theorem. □

Example 1.5.2. A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C?

Solution: We know that the solution of the Newton cooling law equation is

$$T(t) = (T_0 - T_s)\, e^{-kt} + T_s,$$

and we also know that in this case we have

$$T_0 = 45, \qquad T_s = 5, \qquad T(2) = 25.$$

In this example we need to find t_1 such that $T(t_1) = 15$. In order to find that t_1 we first need to find the constant k,

$$T(t) = (45 - 5)\, e^{-kt} + 5 \quad \Rightarrow \quad T(t) = 40\, e^{-kt} + 5.$$

Now use the fact that $T(2) = 25$ C, that is,

$$20 = T(2) = 40\, e^{-2k} \quad \Rightarrow \quad \ln(1/2) = -2k \quad \Rightarrow \quad k = \frac{1}{2}\ln(2).$$

Having the constant k we can now go on and find the time t_1 such that $T(t_1) = 15$ C.

$$T(t) = 40\, e^{-t\ln(\sqrt{2})} + 5 \quad \Rightarrow \quad 10 = 40\, e^{-t_1\ln(\sqrt{2})} \quad \Rightarrow \quad t_1 = 4.$$

$\lhd$

1.5.4. Mixing Problems. We study the system pictured in Fig. 3. A tank has a salt mass $Q(t)$ dissolved in a volume $V(t)$ of water at a time t. Water is pouring into the tank at a rate $r_i(t)$ with a salt concentration $q_i(t)$. Water is also leaving the tank at a rate $r_o(t)$ with a salt concentration $q_o(t)$. Recall that a water rate r means water volume per unit time, and a salt concentration q means salt mass per unit volume.

We assume that the salt entering in the tank gets instantaneously mixed. As a consequence the salt concentration in the tank is homogeneous at every time. This property simplifies the mathematical model describing the salt in the tank.

Before stating the problem we want to solve, we review the physical units of the main fields involved in it. Denote by $[r_i]$ the units of the quantity r_i. Then we have

$$[r_i] = [r_o] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i] = [q_o] = \frac{\text{Mass}}{\text{Volume}},$$
$$[V] = \text{Volume}, \quad [Q] = \text{Mass}.$$

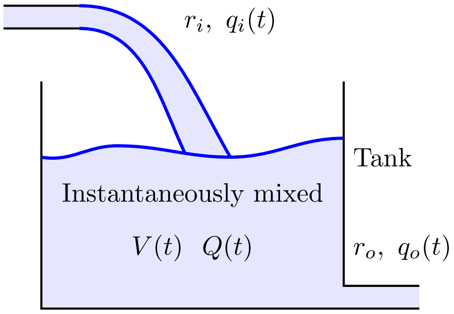


FIGURE 3. Description of a water tank problem.

Definition 1.5.7. *A* ***Mixing Problem*** *refers to water coming into a tank at a rate r_i with salt concentration q_i, and going out the tank at a rate r_o and salt concentration q_o, so that the water volume V and the total amount of salt Q, which is instantaneously mixed, in the tank satisfy the following equations,*

$$V'(t) = r_i(t) - r_o(t), \tag{1.5.1}$$

$$Q'(t) = r_i(t)\, q_i(t) - r_o(t), q_o(t), \tag{1.5.2}$$

$$q_o(t) = \frac{Q(t)}{V(t)}, \tag{1.5.3}$$

$$r_i'(t) = r_o'(t) = 0. \tag{1.5.4}$$

The first and second equations above are just the mass conservation of water and salt, respectively. Water volume and mass are proportional, so both are conserved, and we chose the volume to write down this conservation in Eq. (1.5.1). This equation is indeed a conservation because it says that the water volume variation in time is equal to the difference of volume time rates coming in and going out of the tank. Eq. (1.5.2) is the salt mass conservation, since the salt mass variation in time is equal to the difference of the salt mass time rates coming in and going out of the tank. The product of a water rate r times a salt concentration q has units of mass per time and represents the amount of salt entering or leaving the tank per unit time. Eq. (1.5.3) is the consequence of the instantaneous mixing mechanism in the tank. Since the salt in the tank is well-mixed, the salt concentration is homogeneous in the tank, with value $Q(t)/V(t)$. Finally the equations in (1.5.4) say that both rates in and out are time independent, hence constants.

Theorem 1.5.8. *The amount of salt in the mixing problem above satisfies the equation*

$$Q'(t) = a(t)\,Q(t) + b(t), \tag{1.5.5}$$

where the coefficients in the equation are given by

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\,q_i(t). \tag{1.5.6}$$

Proof of Theorem 1.5.8: The equation for the salt in the tank given in (1.5.5) comes from Eqs. (1.5.1)-(1.5.4). We start noting that Eq. (1.5.4) says that the water rates are constant. We denote them as r_i and r_o. This information in Eq. (1.5.1) implies that V' is constant. Then we can easily integrate this equation to obtain

$$V(t) = (r_i - r_o)\,t + V_0, \tag{1.5.7}$$

where $V_0 = V(0)$ is the water volume in the tank at the initial time $t = 0$. On the other hand, Eqs.(1.5.2) and (1.5.3) imply that

$$Q'(t) = r_i\,q_i(t) - \frac{r_o}{V(t)}\,Q(t).$$

Since $V(t)$ is known from Eq. (1.5.7), we get that the function Q must be solution of the differential equation

$$Q'(t) = r_i\,q_i(t) - \frac{r_o}{(r_i - r_o)\,t + V_0}\,Q(t).$$

This is a linear ODE for the function Q. Indeed, introducing the functions

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\,q_i(t),$$

the differential equation for Q has the form

$$Q'(t) = a(t)\,Q(t) + b(t).$$

This establishes the Theorem. □

We could use the formula for the general solution of a linear equation given in Section 1.2 to write the solution of Eq. (1.5.5) for Q. Such formula covers all cases we are going to study in this section. Since we already know that formula, we choose to find solutions in particular cases. These cases are given by specific choices of the rate constants r_i, r_o, the concentration function q_i, and the initial data constants V_0 and $Q_0 = Q(0)$. The study of solutions to Eq. (1.5.5) in several particular cases might provide a deeper understanding of the physical situation under study than the expression of the solution Q in the general case.

Example 1.5.3 (General Case for $V(t) = V_0$)**.** Consider a mixing problem with equal constant water rates $r_i = r_o = r$, with constant incoming concentration q_i, and with a given initial water volume in the tank V_0. Then, find the solution to the initial value problem

$$Q'(t) = a(t)\,Q(t) + b(t), \qquad Q(0) = Q_0,$$

where function a and b are given in Eq. (1.5.6). Graph the solution function Q for different values of the initial condition Q_0.

Solution: The assumption $r_i = r_o = r$ implies that the function a is constant, while the assumption that q_i is constant implies that the function b is also constant too,

$$\begin{aligned} a(t) &= -\frac{r_o}{(r_i - r_o)\,t + V_0} &&\Rightarrow && a(t) = -\frac{r}{V_0} = a_0,\\ b(t) &= r_i\,q_i(t) &&\Rightarrow && b(t) = r_i\,q_i = b_0. \end{aligned}$$

Then, we must solve the initial value problem for a constant coefficients linear equation,

$$Q'(t) = a_0\,Q(t) + b_0, \qquad Q(0) = Q_0,$$

The integrating factor method can be used to find the solution of the initial value problem above. The formula for the solution is given in Theorem 1.1.4,

$$Q(t) = \Big(Q_0 + \frac{b_0}{a_0}\Big)\,e^{a_0 t} - \frac{b_0}{a_0}.$$

In our case the we can evaluate the constant b_0/a_0, and the result is

$$\frac{b_0}{a_0} = (rq_i)\Big(-\frac{V_0}{r}\Big) \quad \Rightarrow \quad -\frac{b_0}{a_0} = q_i V_0.$$

Then, the solution Q has the form,

$$Q(t) = \big(Q_0 - q_i V_0\big)\,e^{-rt/V_0} + q_i V_0. \tag{1.5.8}$$

The initial amount of salt Q_0 in the tank can be any non-negative real number. The solution behaves differently for different values of Q_0. We classify these values in three classes:

(a) The initial amount of salt in the tank is the critical value $Q_0 = q_i V_0$. In this case the solution Q remains constant equal to this critical value, that is, $Q(t) = q_i V_0$.

(b) The initial amount of salt in the tank is bigger than the critical value, $Q_0 > q_i V_0$. In this case the salt in the tank Q decreases exponentially towards the critical value.

(c) The initial amount of salt in the tank is smaller than the critical value, $Q_0 < q_i V_0$. In this case the salt in the tank Q increases exponentially towards the critical value.

The graphs of a few solutions in these three classes are plotted in Fig. 4.

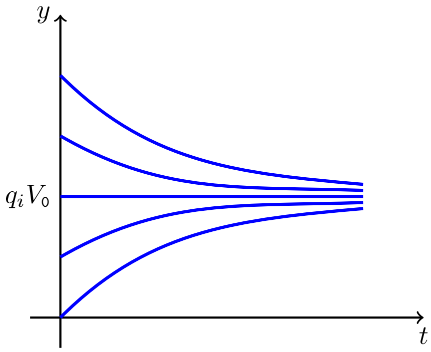


FIGURE 4. The function Q in (1.5.8) for a few values of the initial condition Q_0.

◁

Example 1.5.4 (Find a particular time, for $V(t) = V_0$)**.** Consider a mixing problem with equal constant water rates $r_i = r_o = r$ and fresh water is coming into the tank, hence $q_i = 0$. Then, find the time t_1 such that the salt concentration in the tank $Q(t)/V(t)$ is 1% the initial value. Write that time t_1 in terms of the rate r and initial water volume V_0.

Solution: The first step to solve this problem is to find the solution Q of the initial value problem

$$Q'(t) = a(t)\, Q(t) + b(t), \qquad Q(0) = Q_0,$$

where function a and b are given in Eq. (1.5.6). In this case they are

$$\begin{aligned} a(t) &= -\frac{r_o}{(r_i - r_o)\, t + V_0} &\quad \Rightarrow \quad& a(t) = -\frac{r}{V_0}, \\ b(t) &= r_i\, q_i(t) &\quad \Rightarrow \quad& b(t) = 0. \end{aligned}$$

The initial value problem we need to solve is

$$Q'(t) = -\frac{r}{V_0}\, Q(t), \qquad Q(0) = Q_0.$$

From Section 1.1 we know that the solution is given by

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

We can now proceed to find the time t_1. We first need to find the concentration $Q(t)/V(t)$. We already have $Q(t)$ and we now that $V(t) = V_0$, since $r_i = r_o$. Therefore,

$$\frac{Q(t)}{V(t)} = \frac{Q(t)}{V_0} = \frac{Q_0}{V_0}\, e^{-rt/V_0}.$$

The condition that defines t_1 is

$$\frac{Q(t_1)}{V(t_1)} = \frac{1}{100}\, \frac{Q_0}{V_0}.$$

From these two equations above we conclude that

$$\frac{1}{100}\, \frac{Q_0}{V_0} = \frac{Q(t_1)}{V(t_1)} = \frac{Q_0}{V_0}\, e^{-rt_1/V_0}.$$

The time t_1 comes from the equation

$$\frac{1}{100} = e^{-rt_1/V_0} \quad \Leftrightarrow \quad \ln\Big(\frac{1}{100}\Big) = -\frac{rt_1}{V_0} \quad \Leftrightarrow \quad \ln(100) = \frac{rt_1}{V_0}.$$

The final result is given by

$$t_1 = \frac{V_0}{r}\, \ln(100).$$

◁

Example 1.5.5 (Nonzero q_i, for $V(t) = V_0$)**.** Consider a mixing problem with equal constant water rates $r_i = r_o = r$, with only fresh water in the tank at the initial time, hence $Q_0 = 0$ and with a given initial volume of water in the tank V_0. Then find the function salt in the tank Q if the incoming salt concentration is given by the function

$$q_i(t) = 2 + \sin(2t).$$

Solution: We need to find the solution Q to the initial value problem

$$Q'(t) = a(t)\, Q(t) + b(t), \qquad Q(0) = 0,$$

where function a and b are given in Eq. (1.5.6). In this case we have

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \qquad \Rightarrow \qquad a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i\, q_i(t) \qquad \Rightarrow \qquad b(t) = r\,\big[2 + \sin(2t)\big].$$

We are changing the sign convention for a_0 so that $a_0 > 0$. The initial value problem we need to solve is

$$Q'(t) = -a_0\, Q(t) + b(t), \qquad Q(0) = 0.$$

The solution is computed using the integrating factor method and the result is

$$Q(t) = e^{-a_0 t} \int_0^t e^{a_0 s} b(s)\, ds,$$

where we used that the initial condition is $Q_0 = 0$. Recalling the definition of the function b we obtain

$$Q(t) = e^{-a_0 t} \int_0^t e^{a_0 s} \big[2 + \sin(2s)\big]\, ds.$$

This is the formula for the solution of the problem, we only need to compute the integral given in the equation above. This is not straightforward though. We start with the following integral found in an integration table,

$$\int e^{ks} \sin(ls)\, ds = \frac{e^{ks}}{k^2 + l^2}\,\big[k \sin(ls) - l \cos(ls)\big],$$

where k and l are constants. Therefore,

$$\begin{aligned}\int_0^t e^{a_0 s}\big[2 + \sin(2s)\big]\, ds &= \Big[\frac{2}{a_0} e^{a_0 s}\Big]\Big|_0^t + \Big[\frac{e^{a_0 s}}{a_0^2 + 2^2}\,\big[a_0 \sin(2s) - 2\cos(2s)\big]\Big]\Big|_0^t, \\ &= \frac{2}{a_0} q\big(e^{a_0 t} - 1\big) + \frac{e^{a_0 t}}{a_0^2 + 2^2}\,\big[a_0 \sin(2t) - 2\cos(2t)\big] + \frac{2}{a_0^2 + 2^2}.\end{aligned}$$

With the integral above we can compute the solution Q as follows,

$$Q(t) = e^{-a_0 t}\Big[\frac{2}{a_0}\big(e^{a_0 t} - 1\big) + \frac{e^{a_0 t}}{a_0^2 + 2^2}\,\big[a_0 \sin(2t) - 2\cos(2t)\big] + \frac{2}{a_0^2 + 2^2}\Big],$$

recalling that $a_0 = r/V_0$. We rewrite expression above as follows,

$$Q(t) = \frac{2}{a_0} + \Big[\frac{2}{a_0^2 + 2^2} - \frac{2}{a_0}\Big] e^{-a_0 t} + \frac{1}{a_0^2 + 2^2}\,\big[a_0 \sin(2t) - 2\cos(2t)\big]. \tag{1.5.9}$$

◁

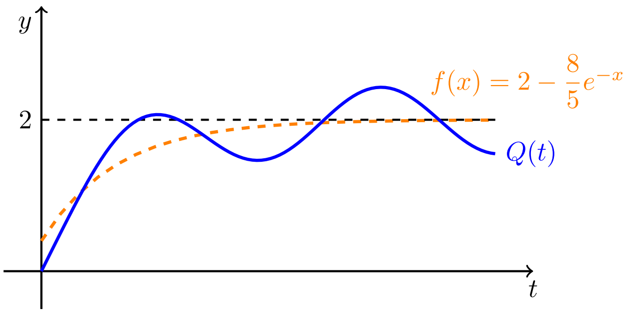


FIGURE 5. The graph of the function Q given in Eq. (1.5.9) for $a_0 = 1$.

1.5.5. Exercises.

1.5.1.- A radioactive material decays at a rate proportional to the amount present. Initially there are 50 milligrams of the material present and after one hour the material has lost 80% of its original mass.

(a) Find the mass of the material as function of time.
(b) Find the mass of the material after four hours.
(c) Find the half-life of the material.

1.5.2.- A vessel with liquid at 18 C is placed in a cooler held at 3 C, and after 3 minutes the temperature drops to 13 C.

(a) Find the differential equation satisfied by the temperature T of a liquid in the cooler at time $t = 0$.
(b) Find the function temperature of the liquid once it is put in the cooler.
(c) Find the liquid cooling constant.

1.5.3.- A tank initially contains $V_0 = 100$ liters of water with $Q_0 = 25$ grams of salt. The tank is rinsed with fresh water flowing in at a rate of $r_i = 5$ liters per minute and leaving the tank at the same rate. The water in the tank is well-stirred. Find the time such that the amount the salt in the tank is $Q_1 = 5$ grams.

1.5.4.- A tank initially contains $V_0 = 100$ liters of pure water. Water enters the tank at a rate of $r_i = 2$ liters per minute with a salt concentration of $q_1 = 3$ grams per liter. The instantaneously mixed mixture leaves the tank at the same rate it enters the tank. Find the salt concentration in the tank at any time $t \geqslant 0$. Also find the limiting amount of salt in the tank in the limit $t \to \infty$.

1.5.5.- A tank with a capacity of $V_m = 500$ liters originally contains $V_0 = 200$ liters of water with $Q_0 = 100$ grams of salt in solution. Water containing salt with concentration of $q_i = 1$ gram per liter is poured in at a rate of $r_i = 3$ liters per minute. The well-stirred water is allowed to pour out the tank at a rate of $r_o = 2$ liters per minute. Find the salt concentration in the tank at the time when the tank is about to overflow. Compare this concentration with the limiting concentration at infinity time if the tank had infinity capacity.

1.6. Nonlinear Equations

Linear differential equations are simpler to solve than nonlinear differential equations. While we have an explicit formula for the solutions to all linear equations—Theorem 1.2.3—there is no such formula for solutions to every nonlinear equation. It is true that we solved several nonlinear equations in §§ 1.2-1.4, and we arrived at different formulas for their solutions, but the nonlinear equations we solved are only a tiny part of all nonlinear equations.

One can give up on the goal of finding a formula for solutions to all nonlinear equations. Then, one can focus on proving whether a nonlinear equations has solutions or not. This is the path followed to arrive at the Picard-Lindelöf Theorem. This theorem determines what nonlinear equations have solutions, but it provides no formula for them. However, the proof of the theorem does provide a way to compute a sequence of approximate solutions to the differential equation. The proof ends showing that this sequence converges to a solution of the differential equation.

In this section we first introduce the Picard-Lindelöf Theorem and the Picard iteration to find approximate solutions. We then compare what we know about solutions to linear and to nonlinear differential equations. We finish this section with a brief discussion on direction fields.

1.6.1. The Picard-Lindelöf Theorem.

We will show that a large class of nonlinear differential equations have solutions. First, let us recall the definition of a nonlinear equation.

Definition 1.6.1. *An ordinary differential equation $y'(t) = f(t, y(t))$ is called **nonlinear** iff the function f is nonlinear in the second argument.*

Example 1.6.1.

(a) The differential equation

$$y'(t) = \frac{t^2}{y^3(t)}$$

is nonlinear, since the function $f(t, y) = t^2/y^3$ is nonlinear in the second argument.

(b) The differential equation

$$y'(t) = 2ty(t) + \ln\big(y(t)\big)$$

is nonlinear, since the function $f(t, y) = 2ty + \ln(y)$ is nonlinear in the second argument, due to the term $\ln(y)$.

(c) The differential equation

$$\frac{y'(t)}{y(t)} = 2t^2$$

is linear, since the function $f(t, y) = 2t^2 y$ is linear in the second argument.

◁

The Picard-Lindelöf Theorem shows that certain nonlinear equations have solutions, uniquely determined by appropriate initial data.

Theorem 1.6.2 (Picard-Lindelöf)**.** *Consider the initial value problem*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1.6.1}$$

If the function f is continuous on the domain $D_a = [t_0 - a, t_0 + a] \times [y_0 - a, y_0 + a] \subset \mathbb{R}^2$, for some $a > 0$, and f is Lipschitz continuous on y, that is there exists $k > 0$ such that

$$|f(t, y_2) - f(t, y_1)| < k\,|y_2 - y_1|,$$

for all (t, y_2), $(t, y_1) \in D_a$, *then there exists a positive* $b < a$ *such that there exists a unique solution* y, *on the domain* $[t_0 - b, t_0 + b]$, *to the initial value problem in (1.6.1).*

Remark: We prove this theorem rewriting the differential equation as an integral equation for the unknown function y. Then we use this integral equation to construct a sequence of approximate solutions $\{y_n\}$ to the original initial value problem. Next we show that this sequence of approximate solutions has a unique limit as $n \to \infty$. We end the proof showing that this limit is the only solution of the original initial value problem. This proof follows [15] § 1.6 and Zeidler's [16] § 1.8. It is important to read the review on complete normed vector spaces, called Banach spaces, given in these references.

Proof of Theorem 1.6.2: We start writing the differential equation in 1.6.1 as an integral equation, hence we integrate on both sides of that equation with respect to t,

$$\int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} f(s, y(s))\,ds \quad \Rightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds. \tag{1.6.2}$$

We have used the Fundamental Theorem of Calculus on the left-hand side of the first equation to get the second equation. And we have introduced the initial condition $y(t_0) = y_0$. We use this integral form of the original differential equation to construct a sequence of functions $\{y_n\}_{n=0}^{\infty}$. The domain of every function in this sequence is $D_a = [t_0 - a, t_0 + a]$. The sequence is defined as follows,

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds, \qquad n \geqslant 0, \qquad y_0(t) = y_0. \tag{1.6.3}$$

We see that the first element in the sequence is the constant function determined by the initial conditions in (1.6.1). The iteration in (1.6.3) is called the Picard iteration. The central idea of the proof is to show that the sequence $\{y_n\}$ is a Cauchy sequence in the space $C(D_b)$ of uniformly continuous functions in the domain $D_b = [t_0 - b, t_0 + b]$ for a small enough $b > 0$. This function space is a Banach space under the norm

$$\|u\| = \max_{t \in D_b} |u(t)|.$$

See [15] and references therein for the definition of Cauchy sequences, Banach spaces, and the proof that $C(D_b)$ with that norm is a Banach space. We now show that the sequence $\{y_n\}$ is a Cauchy sequence in that space. Any two consecutive elements in the sequence satisfy

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \max_{t \in D_b} \Big| \int_{t_0}^{t} f(s, y_n(s))\,ds - \int_{t_0}^{t} f(s, y_{n-1}(s))\,ds \Big| \\
&\leqslant \max_{t \in D_b} \int_{t_0}^{t} \big| f(s, y_n(s)) - f(s, y_{n-1}(s)) \big|\,ds \\
&\leqslant k \max_{t \in D_b} \int_{t_0}^{t} |y_n(s) - y_{n-1}(s)|\,ds \\
&\leqslant kb\,\|y_n - y_{n-1}\|.
\end{aligned}$$

Denoting $r = kb$, we have obtained the inequality

$$\|y_{n+1} - y_n\| \leqslant r\,\|y_n - y_{n-1}\| \quad \Rightarrow \quad \|y_{n+1} - y_n\| \leqslant r^n\,\|y_1 - y_0\|.$$

Using the triangle inequality for norms and and the sum of a geometric series one compute the following,

$$\begin{aligned}
\|y_n - y_{n+m}\| &= \|y_n - y_{n+1} + y_{n+1} - y_{n+2} + \cdots + y_{n+(m-1)} - y_{n+m}\| \\
&\leqslant \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + \cdots + \|y_{n+(m-1)} - y_{n+m}\| \\
&\leqslant (r^n + r^{n+1} + \cdots + r^{n+m})\, \|y_1 - y_0\| \\
&\leqslant r^n (1 + r + r^2 + \cdots + r^m)\, \|y_1 - y_0\| \\
&\leqslant r^n \Big(\frac{1 - r^m}{1 - r}\Big)\, \|y_1 - y_0\|.
\end{aligned}$$

Now choose the positive constant b such that $b < \min\{a, 1/k\}$, hence $0 < r < 1$. In this case the sequence $\{y_n\}$ is a Cauchy sequence in the Banach space $C(D_b)$, with norm $\|\ \|$, hence converges. Denote the limit by $y = \lim_{n\to\infty} y_n$. This function satisfies the equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds,$$

which says that y is not only continuous but also differentiable in the interior of D_b, hence y is solution of the initial value problem in (1.6.1). The proof of uniqueness of the solution follows the same argument used to show that the sequence above is a Cauchy sequence. Consider two solutions y and $\tilde{y}$ of the initial value problem above. That means,

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)\, ds, \qquad \tilde{y}(t) = y_0 + \int_{t_0}^{t} f(s, \tilde{y}(s)\, ds.$$

Therefore, their difference satisfies

$$\begin{aligned}
\|y - \tilde{y}\| &= \max_{t\in D_b} \Big| \int_{t_0}^{t} f(s, y(s))\, ds - \int_{t_0}^{t} f(s, \tilde{y}(s))\, ds \Big| \\
&\leqslant \max_{t\in D_b} \int_{t_0}^{t} \big| f(s, y(s)) - f(s, \tilde{y}(s)) \big|\, ds \\
&\leqslant k \max_{t\in D_b} \int_{t_0}^{t} |y(s) - \tilde{y}(s)|\, ds \\
&\leqslant kb\, \|y - \tilde{y}\|.
\end{aligned}$$

Since b is chosen so that $r = kb < 1$, we got that

$$\|y - \tilde{y}\| \leqslant r\, \|y - \tilde{y}\|, \quad r < 1 \quad \Rightarrow \quad \|y - \tilde{y}\| = 0 \quad \Rightarrow \quad y = \tilde{y}.$$

This establishes the Theorem. □

Example 1.6.2. Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\, y + 3 \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\, ds = \int_0^t (2\, y(s) + 3)\, ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t (2\, y(s) + 3)\, ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t (2\, y(s) + 3)\, ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t \big(2\,y_n(s) + 3\big)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence. We said $y_0 = 1$, now y_1 is given by

$$n = 0, \quad y_1(t) = 1 + \int_0^t \big(2\,y_0(s) + 3\big)\,ds = 1 + \int_0^t 5\,ds = 1 + 5t.$$

So $y_1 = 1 + 5t$. Now we compute y_2,

$$y_2 = 1+\int_0^t \big(2\,y_1(s)+3\big)\,ds = 1+\int_0^t \big(2(1+5s)+3\big)\,ds \quad \Rightarrow \quad y_2 = 1+\int_0^t \big(5+10s\big)\,ds = 1+5t+5t^2.$$

So we've got $y_2(t) = 1 + 5t + 5t^2$. Now y_3,

$$y_3 = 1 + \int_0^t \big(2\,y_2(s) + 3\big)\,ds = 1 + \int_0^t \big(2(1 + 5s + 5s^2) + 3\big)\,ds$$

so we have,

$$y_3 = 1 + \int_0^t \big(5 + 10s + 10s^2\big)\,ds = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3.$$

So we obtained $y_3(t) = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3$. We now rewrite this expression so we can get a power series expansion that can be written in terms of simple functions. The first step is done already, to write the powers of t as t^n, for $n = 1, 2, 3$,

$$y_3(t) = 1 + 5t^1 + 5t^2 + \frac{5(2)}{3}\,t^3$$

We now multiply by one each term so we get the factorials $n!$ on each term

$$y_3(t) = 1 + 5\,\frac{t^1}{1!} + 5(2)\,\frac{t^2}{2!} + 5(2^2)\,\frac{t^3}{3!}$$

We then realize that we can rewrite the expression above in terms of power of $(2t)$, that is,

$$y_3(t) = 1 + \frac{5}{2}\,\frac{(2t)^1}{1!} + \frac{5}{2}\,\frac{(2t)^2}{2!} + \frac{5}{2}\,\frac{(2t)^3}{3!} = 1 + \frac{5}{2}\Big((2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\Big).$$

From this last expressionis simple to guess the n-th approximation

$$y_N(t) = 1 + \frac{5}{2}\Big((2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots + \frac{(2t)^N}{N!}\Big) = 1 + \frac{5}{2}\sum_{k=1}^{N} \frac{(2t)^k}{k!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(at)^k}{k!} \quad \Rightarrow \quad \sum_{k=1}^{\infty} \frac{(at)^k}{k!} = (e^{at} - 1).$$

Then, the limit $N \to \infty$ is given by

$$y(t) = \lim_{N\to\infty} y_N(t) = 1 + \frac{5}{2}\sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = 1 + \frac{5}{2}\big(e^{2t} - 1\big),$$

One last rewriting of the solution and we obtain

$$y(t) = \frac{5}{2}e^{2t} - \frac{3}{2}.$$

◁

Remark: The differential equation $y' = 2\,y + 3$ is of course linear, so the solution to the initial value problem in Example 1.6.2 can be obtained using the methods in Section 1.1,

$$e^{-2t}\,(y' - 2\,y) = e^{-2t}\,3 \quad \Rightarrow \quad e^{-2t}\,y = -\frac{3}{2}\,e^{-2t} + c \quad \Rightarrow \quad y(t) = c\,e^{2t} - \frac{3}{2};$$

and the initial condition implies

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2} \quad \Rightarrow \quad y(t) = \frac{5}{2}\,e^{2t} - \frac{3}{2}.$$

Example 1.6.3. Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = a\,y + b \qquad y(0) = \hat{y}_0, \qquad a, b \in \mathbb{R}.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t (a\,y(s) + b)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t (a\,y(s) + b)\,ds.$$

Using the initial condition, $y(0) = \hat{y}_0$,

$$y(t) = \hat{y}_0 + \int_0^t (a\,y(s) + b)\,ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = \hat{y}_0, \quad y_{n+1}(t) = \hat{y}_0 + \int_0^t (a\,y_n(s) + b)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence. We said $y_0 = \hat{y}_0$, now y_1 is given by

$$\begin{aligned} n = 0, \quad y_1(t) &= y_0 + \int_0^t (a\,y_0(s) + b)\,ds \\ &= \hat{y}_0 + \int_0^t (a\,\hat{y}_0 + b)\,ds \\ &= \hat{y}_0 + (a\,\hat{y}_0 + b)t. \end{aligned}$$

So $y_1 = \hat{y}_0 + (a\,\hat{y}_0 + b)t$. Now we compute y_2,

$$\begin{aligned} y_2 &= \hat{y}_0 + \int_0^t [a\,y_1(s) + b]\,ds \\ &= \hat{y}_0 + \int_0^t \big[a(\hat{y}_0 + (a\,\hat{y}_0 + b)s) + b\big]\,ds \\ &= \hat{y}_0 + (a\hat{y}_0 + b)t + (a\,\hat{y}_0 + b)\frac{at^2}{2} \end{aligned}$$

So we obtained $y_2(t) = \hat{y}_0 + (a\hat{y}_0 + b)t + (a\,\hat{y}_0 + b)\dfrac{at^2}{2}$. A similar calculation gives us y_3,

$$y_3(t) = \hat{y}_0 + (a\hat{y}_0 + b)t + (a\,\hat{y}_0 + b)\frac{at^2}{2} + (a\,\hat{y}_0 + b)\frac{a^2t^3}{3!}.$$

We now rewrite this expression so we can get a power series expansion that can be written in terms of simple functions. The first step is done already, to write the powers of t as t^n, for $n = 1, 2, 3$,

$$y_3(t) = \hat{y}_0 + (a\hat{y}_0 + b)\,\frac{(t)^1}{1!} + (a\,\hat{y}_0 + b)\,a\,\frac{t^2}{2!} + (a\,\hat{y}_0 + b)\,a^2\,\frac{t^3}{3!}.$$

We already have the factorials $n!$ on each term t^n. We now realize we can write the power functions as $(at)^n$ is we multiply eat term by one, as follows

$$y_3(t) = \hat{y}_0 + \frac{(a\hat{y}_0 + b)}{a}\,\frac{(at)^1}{1!} + \frac{(a\,\hat{y}_0 + b)}{a}\,\frac{(at)^2}{2!} + \frac{(a\,\hat{y}_0 + b)}{a}\,\frac{(at)^3}{3!}.$$

Now we can pull a common factor

$$y_3(t) = \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\,\Big(\frac{(at)^1}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!}\Big)$$

From this last expressionis simple to guess the n-th approximation

$$y_N(t) = \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\,\Big(\frac{(at)^1}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots + \frac{(at)^N}{N!}\Big)$$

$$\lim_{N\to\infty} y_N(t) = \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\sum_{k=1}^{\infty}\frac{(at)^k}{k!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty}\frac{(at)^k}{k!} = 1 + \sum_{k=1}^{\infty}\frac{(at)^k}{k!} \quad\Rightarrow\quad \sum_{k=1}^{\infty}\frac{(at)^k}{k!} = (e^{at} - 1).$$

Notice that the sum in the exponential starts at $k = 0$, while the sum in y_n starts at $k = 1$. Then, the limit $n \to \infty$ is given by

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_n(t) \\ &= \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\sum_{k=1}^{\infty}\frac{(at)^k}{k!} \\ &= \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\,\big(e^{at} - 1\big), \end{aligned}$$

We have been able to add the power series and we have the solution written in terms of simple functions. One last rewriting of the solution and we obtain

$$y(t) = \Big(\hat{y}_0 + \frac{b}{a}\Big)\,e^{at} - \frac{b}{a}.$$

◁

Remark: We reobtained Eq. (1.1.12) in Theorem 1.1.4.

Example 1.6.4. Use the Picard iteration to find the solution of

$$y' = 5t\,y, \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t 5s\,y(s)\,ds \quad\Rightarrow\quad y(t) - y(0) = \int_0^t 5s\,y(s)\,ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t 5s\,y(s)\,ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t 5s\,y_n(s)\,ds, \quad n \geqslant 0.$$

We now compute the first four elements in the sequence. The first one is $y_0 = y(0) = 1$, the second one y_1 is given by

$$n = 0, \quad y_1(t) = 1 + \int_0^t 5s\, ds = 1 + \frac{5}{2}\, t^2.$$

So $y_1 = 1 + (5/2)t^2$. Now we compute y_2,

$$\begin{aligned} y_2 &= 1 + \int_0^t 5s\, y_1(s)\, ds \\ &= 1 + \int_0^t 5s\, \big(1 + \frac{5}{2}\, s^2\big)\, ds \\ &= 1 + \int_0^t \big(5s + \frac{5^2}{2}\, s^3\big)\, ds \\ &= 1 + \frac{5}{2}\, t^2 + \frac{5^2}{8}\, t^4. \end{aligned}$$

So we obtained $y_2(t) = 1 + \frac{5}{2} t^2 + \frac{5^2}{2^3}\, t^4$. A similar calculation gives us y_3,

$$\begin{aligned} y_3 &= 1 + \int_0^t 5s\, y_2(s)\, ds \\ &= 1 + \int_0^t 5s\, \big(1 + \frac{5}{2}\, s^2 + \frac{5^2}{2^3}\, s^4\big)\, ds \\ &= 1 + \int_0^t \big(5s + \frac{5^2}{2}\, s^3 + \frac{5^3}{2^3}\, s^5\big)\, ds \\ &= 1 + \frac{5}{2}\, t^2 + \frac{5^2}{8}\, t^4 + \frac{5^3}{2^3 6}\, t^6. \end{aligned}$$

So we obtained $y_3(t) = 1 + \frac{5}{2} t^2 + \frac{5^2}{2^3}\, t^4 + \frac{5^3}{2^4 3}\, t^6$. We now rewrite this expression so we can get a power series expansion that can be written in terms of simple functions. The first step is to write the powers of t as t^n, for $n = 1, 2, 3$,

$$y_3(t) = 1 + \frac{5}{2} (t^2)^1 + \frac{5^2}{2^3}\, (t^2)^2 + \frac{5^3}{2^4 3}\, (t^2)^3.$$

Now we multiply by one each term to get the right facctorials, $n!$ on each term,

$$y_3(t) = 1 + \frac{5}{2}\, \frac{(t^2)^1}{1!} + \frac{5^2}{2^2}\, \frac{(t^2)^2}{2!} + \frac{5^3}{2^3}\, \frac{(t^2)^3}{3!}.$$

No we realize that the factor $5/2$ can be written together with the powers of t^2,

$$y_3(t) = 1 + \frac{(\frac{5}{2} t^2)}{1!} + \frac{(\frac{5}{2}\, t^2)^2}{2!} + \frac{(\frac{5}{2}\, t^2)^3}{3!}.$$

From this last expression is simple to guess the n-th approximation

$$y_N(t) = 1 + \sum_{k=1}^{N} \frac{(\frac{5}{2} t^2)^k}{k!},$$

which can be proven by induction. Therefore,

$$y(t) = \lim_{N\to\infty} y_N(t) = 1 + \sum_{k=1}^{\infty} \frac{(\frac{5}{2} t^2)^k}{k!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(at)^k}{k!}.$$

so we get

$$y(t) = 1 + (e^{\frac{5}{2}t^2} - 1) \quad \Rightarrow \quad y(t) = e^{\frac{5}{2}t^2}.$$

$\triangleleft$

Remark: The differential equation $y' = 5t\,y$ is of course separable, so the solution to the initial value problem in Example 1.6.4 can be obtained using the methods in Section 1.3,

$$\frac{y'}{y} = 5t \quad \Rightarrow \quad \ln(y) = \frac{5t^2}{2} + c. \quad \Rightarrow \quad y(t) = \tilde{c}\, e^{\frac{5}{2}t^2}.$$

We now use the initial condition,

$$1 = y(0) = \tilde{c} \quad \Rightarrow \quad c = 1,$$

so we obtain the solution

$$y(t) = e^{\frac{5}{2}t^2}.$$

Example 1.6.5. Use the Picard iteration to find the solution of

$$y' = 2t^4\, y, \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\, ds = \int_0^t 2s^4\, y(s)\, ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t 2s^4\, y(s)\, ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t 2s^4\, y(s)\, ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t 2s^4\, y_n(s)\, ds, \quad n \geqslant 0.$$

We now compute the first four elements in the sequence. The first one is $y_0 = y(0) = 1$, the second one y_1 is given by

$$n = 0, \quad y_1(t) = 1 + \int_0^t 2s^4\, ds = 1 + \frac{2}{5}\, t^5.$$

So $y_1 = 1 + (2/5)t^5$. Now we compute y_2,

$$\begin{aligned} y_2 &= 1 + \int_0^t 2s^4\, y_1(s)\, ds \\ &= 1 + \int_0^t 2s^4 \big(1 + \frac{2}{5}\, s^5\big)\, ds \\ &= 1 + \int_0^t \big(2s^4 + \frac{2^2}{5}\, s^9\big)\, ds \\ &= 1 + \frac{2}{5}\, t^5 + \frac{2^2}{5}\, \frac{1}{10} t^{10}. \end{aligned}$$

So we obtained $y_2(t) = 1 + \frac{2}{5} t^5 + \frac{2^2}{5^2}\, \frac{1}{2}\, t^{10}$. A similar calculation gives us y_3,

$$\begin{aligned} y_3 &= 1 + \int_0^t 2s^4\, y_2(s)\, ds \\ &= 1 + \int_0^t 2s^4 \big(1 + \frac{2}{5}\, s^5 + \frac{2^2}{5^2}\, \frac{1}{2} s^{10}\big)\, ds \\ &= 1 + \int_0^t \big(2s^4 + \frac{2^2}{5}\, s^9 + \frac{2^3}{5^2}\, \frac{1}{2}\, s^{14}\big)\, ds \\ &= 1 + \frac{2}{5}\, t^5 + \frac{2^2}{5}\, \frac{1}{10}\, t^{10} + \frac{2^3}{5^2}\, \frac{1}{2}\, \frac{1}{15}\, t^{15}. \end{aligned}$$

So we obtained $y_3(t) = 1 + \frac{2}{5} t^5 + \frac{2^2}{5^2}\, \frac{1}{2}\, t^{10} + \frac{2^3}{5^3}\, \frac{1}{2} \frac{1}{3}\, t^{15}$. We now try reorder terms in this last expression so we can get a power series expansion we can write in terms of simple functions. This is what we do:

$$\begin{aligned} y_3(t) &= 1 + \frac{2}{5} (t^5) + \frac{2^2}{5^3}\, \frac{(t^5)^2}{2} + \frac{2^3}{5^4}\, \frac{(t^5)^3}{6} \\ &= 1 + \frac{2}{5}\, \frac{(t^5)}{1!} + \frac{2^2}{5^2}\, \frac{(t^5)^2}{2!} + \frac{2^3}{5^3}\, \frac{(t^5)^3}{3!} \\ &= 1 + \frac{(\frac{2}{5} t^5)}{1!} + \frac{(\frac{2}{5}\, t^5)^2}{2!} + \frac{(\frac{2}{5}\, t^5)^3}{3!}. \end{aligned}$$

From this last expression is simple to guess the n-th approximation

$$y_N(t) = 1 + \sum_{n=1}^{N} \frac{(\frac{2}{5} t^5)^n}{n!},$$

which can be proven by induction. Therefore,

$$y(t) = \lim_{N \to \infty} y_N(t) = 1 + \sum_{n=1}^{\infty} \frac{(\frac{2}{5} t^5)^n}{n!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(at)^k}{k!}.$$

so we get

$$y(t) = 1 + (e^{\frac{2}{5} t^5} - 1) \quad \Rightarrow \quad y(t) = e^{\frac{2}{5} t^5}.$$

◁

1.6.2. Comparison of Linear and Nonlinear Equations. The main result in § 1.2 was Theorem 1.2.3, which says that an initial value problem for a linear differential equation

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0,$$

with a, b continuous functions on (t_1, t_2), and constants $t_0 \in (t_1, t_2)$ and $y_0 \in \mathbb{R}$, has the unique solution y on (t_1, t_2) given by

$$y(t) = e^{A(t)} \Big(y_0 + \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds \Big),$$

where we introduced the function $A(t) = \displaystyle\int_{t_0}^{t} a(s)\, ds$.

From the result above we can see that solutions to linear differential equations satisfiy the following properties:

(a) There is an explicit expression for the solutions of a differential equations.
(b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution.
(c) For every initial condition $y_0 \in \mathbb{R}$ the solution $y(t)$ is defined for all (t_1, t_2).

Remark: None of these properties hold for solutions of nonlinear differential equations.

From the Picard-Lindelöf Theorem one can see that solutions to nonlinear differential equations satisfy the following properties:

(i) There is no explicit formula for the solution to every nonlinear differential equation.
(ii) Solutions to initial value problems for nonlinear equations may be non-unique when the function f does not satisfy the Lipschitz condition.
(iii) The domain of a solution y to a nonlinear initial value problem may change when we change the initial data y_0.

The next three examples (1.6.6)-(1.6.8) are particular cases of the statements in (i)-(iii). We start with an equation whose *solutions cannot be written in explicit form.*

Example 1.6.6. For every constant a_1, a_2, a_3, a_4, find all solutions y to the equation

$$y'(t) = \frac{t^2}{\big(y^4(t) + a_4\, y^3(t) + a_3\, y^2(t) + a_2\, y(t) + a_1\big)}. \tag{1.6.4}$$

Solution: The nonlinear differential equation above is separable, so we follow § 1.3 to find its solutions. First we rewrite the equation as

$$\big(y^4(t) + a_4\, y^3(t) + a_3\, y^2(t) + a_2\, y(t) + a_1\big)\, y'(t) = t^2.$$

Then we integrate on both sides of the equation,

$$\int \big(y^4(t) + a_4\, y^3(t) + a_3\, y^2(t) + a_2\, y(t) + a_1\big)\, y'(t)\, dt = \int t^2\, dt + c.$$

Introduce the substitution $u = y(t)$, so $du = y'(t)\, dt$,

$$\int \big(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\big)\, du = \int t^2\, dt + c.$$

Integrate the left-hand side with respect to u and the right-hand side with respect to t. Substitute u back by the function y, hence we obtain

$$\frac{1}{5}\, y^5(t) + \frac{a_4}{4}\, y^4(t) + \frac{a_3}{3}\, y^3(t) + \frac{a_2}{2}\, y(t) + a_1\, y(t) = \frac{t^3}{3} + c.$$

This is an implicit form for the solution y of the problem. The solution is the root of a polynomial degree five for all possible values of the polynomial coefficients. But it has been proven that there is no formula for the roots of a general polynomial degree bigger or equal five. We conclude that that there is no explicit expression for solutions y of Eq. (1.6.4). ◁

We now give an example of the statement in (ii), that is, a differential equation which does not satisfy one of the hypothesis in Theorem 1.6.2. The function f has a discontinuity at a line in the (t,u) plane where the initial condition for the initial value problem is given. We then show that *such initial value problem has two solutions* instead of a unique solution.

Example 1.6.7. Find every solution y of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0. \tag{1.6.5}$$

Remark: The equation above is nonlinear, separable, and $f(t,u) = u^{1/3}$ has derivative

$$\partial_u f = \frac{1}{3}\,\frac{1}{u^{2/3}}.$$

Since the function $\partial_u f$ is not continuous at $u = 0$, it does not satisfies the Lipschitz condition in Theorem 1.6.2 on any domain of the form $S = [-a,a] \times [-a,a]$ with $a > 0$.

Solution: The solution to the initial value problem in Eq. (1.6.5) exists but it is not unique, since we now show that it has two solutions. The first solution is

$$y_1(t) = 0.$$

The second solution can be computed as using the ideas from separable equations, that is,

$$\int \big[y(t)\big]^{-1/3}\,y'(t)\,dt = \int dt + c_0.$$

Then, the substitution $u = y(t)$, with $du = y'(t)\,dt$, implies that

$$\int u^{-1/3}\,du = \int dt + c_0.$$

Integrate and substitute back the function y. The result is

$$\frac{3}{2}\big[y(t)\big]^{2/3} = t + c_0 \quad \Rightarrow \quad y(t) = \Big[\frac{2}{3}(t + c_0)\Big]^{3/2}.$$

The initial condition above implies

$$0 = y(0) = \Big(\frac{2}{3}c_0\Big)^{3/2} \quad \Rightarrow \quad c_0 = 0,$$

so the second solution is:

$$y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}.$$

◁

Finally, an example of the statement in (iii). In this example we have an equation with *solutions defined in a domain that depends on the initial data.*

Example 1.6.8. Find the solution y to the initial value problem

$$y'(t) = y^2(t), \qquad y(0) = y_0.$$

Solution: This is a nonlinear separable equation, so we can again apply the ideas in Sect. 1.3. We first find all solutions of the differential equation,

$$\int \frac{y'(t)\,dt}{y^2(t)} = \int dt + c_0 \quad \Rightarrow \quad -\frac{1}{y(t)} = t + c_0 \quad \Rightarrow \quad y(t) = -\frac{1}{c_0 + t}.$$

We now use the initial condition in the last expression above,

$$y_0 = y(0) = -\frac{1}{c_0} \quad \Rightarrow \quad c_0 = -\frac{1}{y_0}.$$

So, the solution of the initial value problem above is:

$$y(t) = \frac{1}{\left(\frac{1}{y_0} - t\right)}.$$

This solution diverges at $t = 1/y_0$, so the domain of the solution y is not the whole real line $\mathbb{R}$. Instead, the domain is $\mathbb{R} - \{y_0\}$, so it depends on the values of the initial data y_0. ◁

In the next example we consider an equation of the form $y'(t) = f(t, y(t))$, where f does not satisfy the hypotheses in Theorem 1.6.2.

Example 1.6.9. Consider the nonlinear initial value problem

$$y'(t) = \frac{1}{(t-1)(t+1)(y(t)-2)(y(t)+3)},$$
$$y(t_0) = y_0. \tag{1.6.6}$$

Find the regions on the plane where the hypotheses in Theorem 1.6.2 are not satisfied.

Solution: In this case the function f is given by:

$$f(t,u) = \frac{1}{(t-1)(t+1)(u-2)(u+3)}, \tag{1.6.7}$$

so f is not defined on the lines

$$t = 1, \qquad t = -1, \qquad u = 2, \qquad u = -3.$$

See Fig. 6. For example, in the case that the initial data is $t_0 = 0$, $y_0 = 1$, then Theorem 1.6.2 implies that there exists a unique solution on any region $\hat{R}$ contained in the rectangle $R = (-1,1) \times (-3,2)$. If the initial data for the initial value problem in Eq. (1.6.6) is $t = 0$, $y_0 = 2$, then the hypotheses of Theorem 1.6.2 are not satisfied. ◁

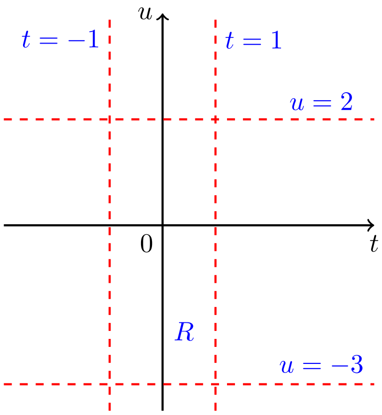


FIGURE 6. Red regions where f in Eq. (1.6.7) is not defined.

1.6.3. Direction Fields. Sometimes one needs to find information about solutions of a differential equation without having to actually solve the equation. One way to do this is with the direction fields. Consider a differential equation

$$y'(t) = f(t, y(t)).$$

We interpret the the right-hand side above in a new way.

(a) In the usual way, the graph of f is a surface in the tyz-space, where $z = f(t,y)$,
(b) In the new way, $f(t,y)$ is the value of a slope of a segment at each point (t,y) on the ty-plane.
(c) That slope is the value of $y'(t)$, the derivative of a solution y at t.

The ideas above suggest the following definition.

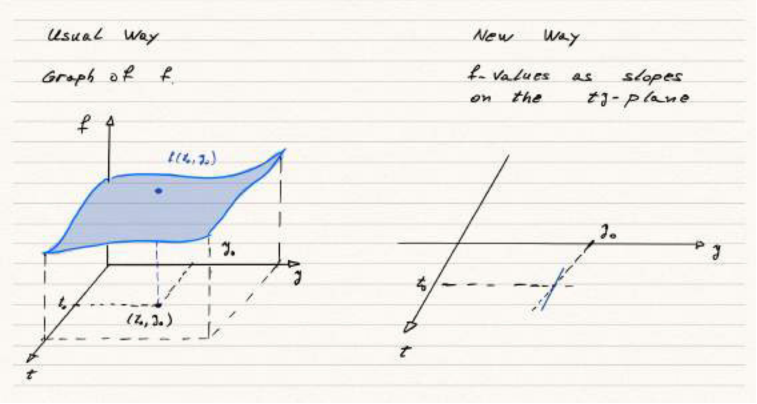


FIGURE 7. The function f as a slope of a segment.

Definition 1.6.3. *A* ***direction field*** *for the differential equation* $y'(t) = f(t, y(t))$ *is the graph on the ty-plane of the values* $f(t, y)$ *as slopes of a small segments.*

We now show the direction fields of e few equations.

Example 1.6.10. Find the direction field of the equation $y' = y$, and sketch a few solutions to the differential equation for different initial conditions.

Solution: Recall that the solutions are $y(t) = y_0\, e^t$. So is the direction field shown in Fig. 8. ◁

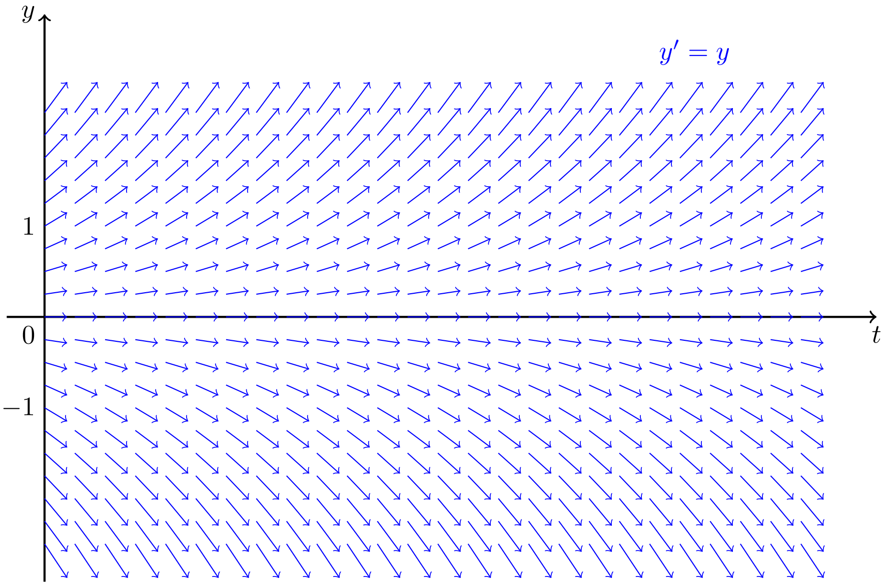


FIGURE 8. Direction field for the equation $y' = y$.

Example 1.6.11. Find the direction field of the equation $y' = \sin(y)$, and sketch a few solutions to the differential equation for different initial conditions.

Solution: The equation is separable so the solutions are

$$\ln\left|\frac{\csc(y_0) + \cot(y_0)}{\csc(y) + \cot(y)}\right| = t,$$

for any $y_0 \in \mathbb{R}$. The graphs of these solutions are not simple to do. But the direction field is simpler to plot and can be seen in Fig. 9. ◁

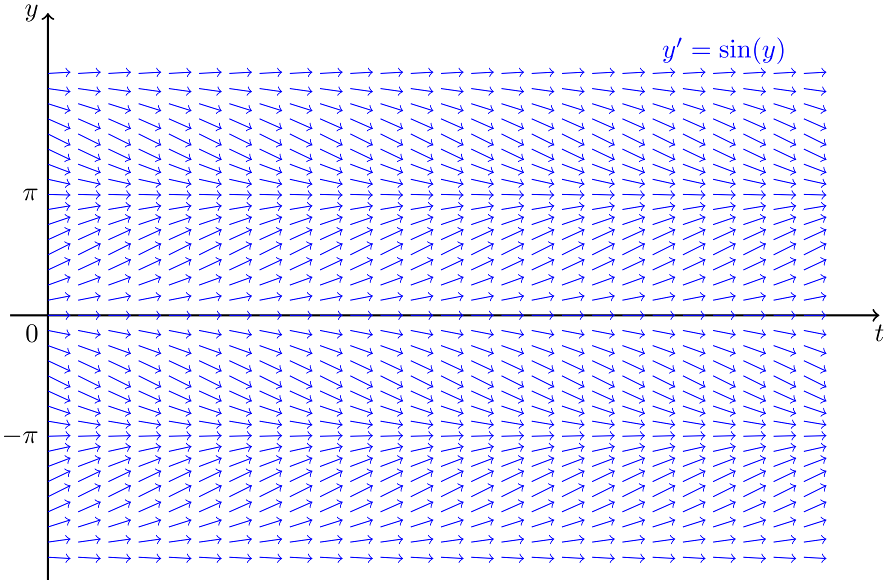


FIGURE 9. Direction field for the equation $y' = \sin(y)$.

Example 1.6.12. Find the direction field of the equation $y' = 2\cos(t)\cos(y)$, and sketch a few solutions to the differential equation for different initial conditions.

Solution: We do not need to compute the explicit solution of $y' = 2\cos(t)\cos(y)$ to have a qualitative idea of its solutions. The direction field can be seen in Fig. 10. ◁

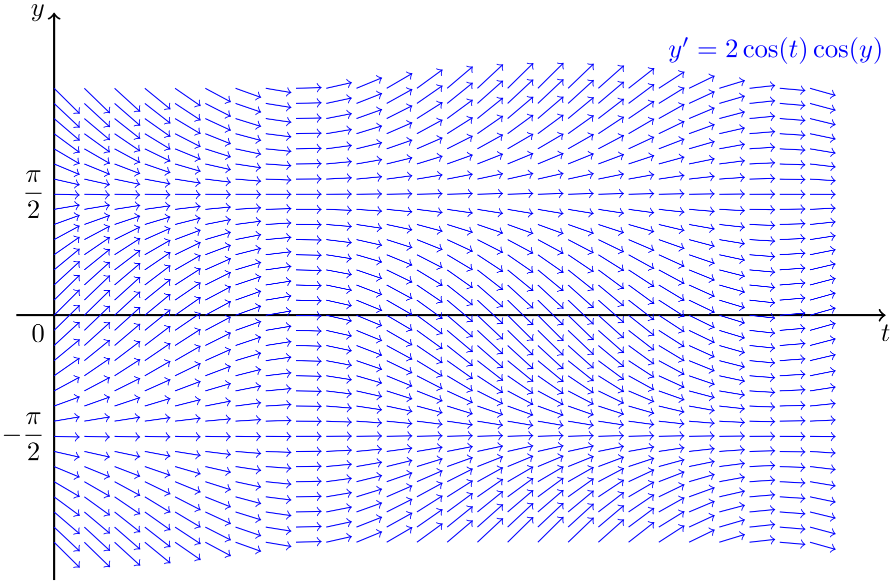


FIGURE 10. Direction field for the equation $y' = 2\cos(t)\cos(y)$.

1.6.4. Exercises.

1.6.1.- Use the Picard iteration to find the first four elements, y_0, y_1, y_2, and y_3, of the sequence $\{y_n\}_{n=0}^{\infty}$ of approximate solutions to the initial value problem

$$y' = 6y + 1, \quad y(0) = 0.$$

1.6.2.- Use the Picard iteration to find the information required below about the sequence $\{y_n\}_{n=0}^{\infty}$ of approximate solutions to the initial value problem

$$y' = 3y + 5, \quad y(0) = 1.$$

(a) The first 4 elements in the sequence, y_0, y_1, y_2, and y_3.

(b) The general term $c_k(t)$ of the approximation

$$y_n(t) = 1 + \sum_{k=1}^{n} \frac{c_k(t)}{k!}.$$

(c) Find the limit $y(t) = \lim_{n\to\infty} y_n(t)$.

1.6.3.- Find the domain where the solution of the initial value problems below is well-defined.

(a) $y' = \dfrac{-4t}{y}, \quad y(0) = y_0 > 0.$

(b) $y' = 2ty^2, \quad y(0) = y_0 > 0.$

1.6.4.- By looking at the equation coefficients, find a domain where the solution of the initial value problem below exists,

(a) $(t^2 - 4)\, y' + 2\ln(t)\, y = 3t$, and initial condition $y(1) = -2$.

(b) $y' = \dfrac{y}{t(t-3)}$, and initial condition $y(-1) = 2$.

1.6.5.- State where in the plane with points (t, y) the hypothesis of Theorem 1.6.2 are not satisfied.

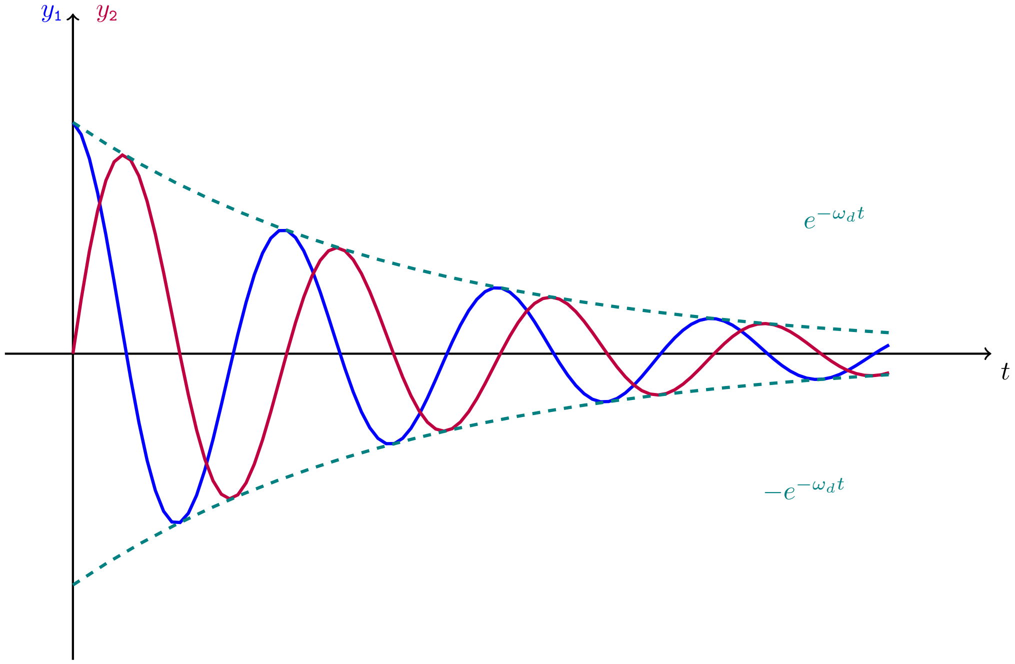
(a) $y' = \dfrac{y^2}{2t - 3y}$.

(b) $y' = \sqrt{1 - t^2 - y^2}$.

CHAPTER 2

Second Order Linear Equations

Newton's second law of motion, $ma = f$, is maybe one of the first differential equations written. This is a second order equation, since the acceleration is the second time derivative of the particle position function. Second order differential equations are more difficult to solve than first order equations. In § 2.1 we compare results on linear first and second order equations. While there is an explicit formula for all solutions to first order linear equations, not such formula exists for all solutions to second order linear equations. The most one can get is the result in Theorem 2.1.7. In § 2.2 we introduce the Reduction Order Method to find a new solution of a second order equation if we already know one solution of the equation. In § 2.3 we find explicit formulas for all solutions to linear second order equations that are both homogeneous and with constant coefficients. These formulas are generalized to nonhomogeneous equations in § 2.5. In § 2.6 we describe a few physical systems described by second order linear differential equations.



2.1. Variable Coefficients

We studied first order linear equations in § 1.1-1.2, where we obtained a formula for all solutions to these equations. We could say that we know all that can be known about solutions to *first order linear* equations. However, this is not the case for solutions to second order linear equations, since we do not have a general formula for all solutions to these equations.

In this section we present two main results, the first one is Theorem 2.1.2, which says that there are solutions to second order linear equations when the equation coefficients are continuous functions. Furthermore, these solutions have two free parameters that can be fixed by appropriate initial conditions.

The second result is Theorem 2.1.7, which is the closest we can get to a formula for solutions to second order linear equations without sources—homogeneous equations. To know all solutions to these equations we only need to know two solutions that are not proportional to each other. The proof of Theorem 2.1.7 is based on Theorem 2.1.2 plus an algebraic calculation and properties of the Wronskian function, which are derived from Abel's Theorem.

2.1.1. Definitions and Examples.

We start with a definition of second order linear differential equations. After a few examples we state the first of the main results, Theorem 2.1.2, about existence and uniqueness of solutions to an initial value problem in the case that the equation coefficients are continuous functions.

Definition 2.1.1. *A* ***second order linear differential equation*** *for the function y is*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \tag{2.1.1}$$

where a_1, a_0, b are given functions on the interval $I \subset \mathbb{R}$. The Eq. (2.1.1) above:

(a) is ***homogeneous*** *iff the source $b(t) = 0$ for all $t \in \mathbb{R}$;*
(b) has ***constant coefficients*** *iff a_1 and a_0 are constants;*
(c) has ***variable coefficients*** *iff either a_1 or a_0 is not constant.*

Remark: The notion of an homogeneous equation presented here is different from the Euler homogeneous equations we studied in § 1.3.

Example 2.1.1.

(a) A second order, linear, homogeneous, constant coefficients equation is

$$y'' + 5y' + 6 = 0.$$

(b) A second order, linear, nonhomogeneous, constant coefficients, equation is

$$y'' - 3y' + y = \cos(3t).$$

(c) A second order, linear, nonhomogeneous, variable coefficients equation is

$$y'' + 2t\, y' - \ln(t)\, y = e^{3t}.$$

(d) Newton's law of motion for a point particle of mass m moving in one space dimension under a force f is mass times acceleration equals force,

$$m\, y''(t) = f(t, y(t), y'(t)).$$

(e) Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

$$-\frac{\hbar^2}{2m}\, \psi'' + V(x)\, \psi = E\, \psi,$$

where ψ is the probability density of finding a particle of mass m at the position x having energy E under a potential V, where $\hbar$ is Planck constant divided by 2π. ◁

Example 2.1.2. Find the differential equation satisfied by the family of functions

$$y(t) = c_1\, e^{4t} + c_2\, e^{-4t},$$

where c_1, c_2 are arbitrary constants.

Solution: From the definition of y compute c_1,

$$c_1 = y\, e^{-4t} - c_2\, e^{-8t}.$$

Now compute the derivative of function y

$$y' = 4c_1\, e^{4t} - 4c_2\, e^{-4t},$$

Replace c_1 from the first equation above into the expression for y',

$$y' = 4(y\, e^{-4t} - c_2\, e^{-8t})e^{4t} - 4c_2\, e^{-4t} \quad \Rightarrow \quad y' = 4y + (-4-4)c_2\, e^{-4t},$$

so we get an expression for c_2 in terms of y and y',

$$y' = 4y - 8c_2\, e^{-4t} \quad \Rightarrow \quad c_2 = \frac{1}{8}(4y - y')\, e^{4t}$$

At this point we can compute c_1 in terms of y and y', although we do not need it for what follows. Anyway,

$$c_1 = y\, e^{-4t} - \frac{1}{8}(4y - y')e^{4t}e^{-8t} \quad \Rightarrow \quad c_1 = \frac{1}{8}\,(4y + y')\, e^{-4t}.$$

We do not need c_1 because we can get a differential equation for y from the equation for c_2. Compute the derivative of that equation,

$$0 = c_2' = \frac{1}{2}(4y - y')\, e^{4t} + \frac{1}{8}(4y' - y'')\, e^{4t} \quad \Rightarrow \quad 4(4y - y') + (4y' - y'') = 0$$

which gives us the following second order linear differential equation for y,

$$y'' - 16\, y = 0.$$

◁

Example 2.1.3. Find the differential equation satisfied by the family of functions

$$y(t) = \frac{c_1}{t} + c_2\, t, \qquad c_1, c_2 \in \mathbb{R}.$$

Solution: Compute $y' = -\dfrac{c_1}{t^2} + c_2$. Get one constant from y' and put it in y,

$$c_2 = y' + \frac{c_1}{t^2} \quad \Rightarrow \quad y = \frac{c_1}{t} + \Big(y' + \frac{c_1}{t^2}\Big)\, t,$$

so we get

$$y = \frac{c_1}{t} + t\, y' + \frac{c_1}{t} \quad \Rightarrow \quad y = \frac{2c_1}{t} + t\, y'.$$

Compute the constant from the expression above,

$$\frac{2c_1}{t} = y - t\, y' \quad \Rightarrow \quad 2c_1 = t\, y - t^2\, y'.$$

Since the left hand side is constant,

$$0 = (2c_1)' = (t\, y - t^2\, y')' = y + t\, y' - 2t\, y' - t^2\, y'',$$

so we get that y must satisfy the differential equation

$$t^2\, y'' + t\, y' - y = 0.$$

$\lhd$

Example 2.1.4. Find the differential equation satisfied by the family of functions

$$y(x) = c_1\, x + c_2\, x^2,$$

where c_1, c_2 are arbitrary constants.

Solution: Compute the derivative of function y

$$y'(x) = c_1 + 2c_2\, x,$$

From here it is simple to get c_1,

$$c_1 = y' - 2c_2\, x.$$

Use this expression for c_1 in the expression for y,

$$y = (y' - 2c_2\, x)\, x + c_2\, x^2 = x\, y' - c_2\, x^2 \quad \Rightarrow \quad c_2 = \frac{y'}{x} - \frac{y}{x^2}.$$

To get the differential equation for y we do not need c_1, but we compute it anyway,

$$c_1 = y' - 2\Big(\frac{y'}{x} - \frac{y}{x^2}\Big)\, x = y' - 2\, y' + \frac{2y}{x} \quad \Rightarrow \quad c_1 = -y' + \frac{2y}{x}.$$

The equation for y can be obtained computing a derivative in the expression for c_2,

$$0 = c_2' = \frac{y''}{x} - \frac{y'}{x^2} - \frac{y'}{x^2} + 2\frac{y}{x^3} = \frac{y''}{x} - 2\frac{y'}{x^2} + 2\frac{y}{x^3} = 0 \quad \Rightarrow \quad x^2\, y'' - 2x\, y' + 2\, y = 0.$$

$\lhd$

2.1.2. Solutions to the Initial Value Problem.

Here is the first of the two main results in this section. Second order linear differential equations have solutions in the case that the equation coefficients are continuous functions. Since the solution is unique when we specify two initial conditions, the general solution must have two arbitrary integration constants.

Theorem 2.1.2 (IVP). *If the functions a_1, a_0, b are continuous on a closed interval $I \subset \mathbb{R}$, the constant $t_0 \in I$, and y_0, $y_1 \in \mathbb{R}$ are arbitrary constants, then there is a unique solution y, defined on I, of the initial value problem*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{2.1.2}$$

Remark: The fixed point argument used in the proof of Picard-Lindelöf's Theorem 1.6.2 can be extended to prove Theorem 2.1.2.

Example 2.1.5. Find the domain of the solution to the initial value problem

$$(t-1)\, y'' - 3t\, y' + \frac{4(t-1)}{(t-3)}\, y = t(t-1), \qquad y(2) = 1, \qquad y'(2) = 0.$$

Solution: We first write the equation above in the form given in the Theorem above,

$$y'' - \frac{3t}{(t-1)}\, y' + \frac{4}{(t-3)}\, y = t.$$

The equation coefficients are defined on the domain

$$(-\infty, 1) \cup (1, 3) \cup (3, \infty).$$

So the solution may not be defined at $t = 1$ or $t = 3$. That is, the solution is defined in

$$(-\infty, 1) \qquad \text{or} \qquad (1, 3) \qquad \text{or} \qquad (3, \infty).$$

Since the initial condition is at $t_0 = 2 \in (1, 3)$, then the domain of the solution is

$$D = (1, 3).$$

$\lhd$

2.1.3. Properties of Homogeneous Equations.

We simplify the problem with the hope to get deeper properties of its solutions. From now on in this section we focus on homogeneous equations only. We will get back to non-homogeneous equations in a later section. But before getting into homogeneous equations, we introduce a new notation to write differential equations. This is a shorter, more economical, notation. Given two functions a_1, a_0, introduce the function L acting on a function y, as follows,

$$L(y) = y'' + a_1(t)\, y' + a_0(t)\, y. \tag{2.1.3}$$

The function L acts on the function y and the result is another function, given by Eq. (2.1.3).

Example 2.1.6. Compute the operator $L(y) = t\, y'' + 2y' - \dfrac{8}{t}\, y$ acting on $y(t) = t^3$.

Solution: Since $y(t) = t^3$, then $y'(t) = 3t^2$ and $y''(t) = 6t$, hence

$$L(t^3) = t\,(6t) + 2(3t^2) - \frac{8}{t}\, t^3 \quad \Rightarrow \quad L(t^3) = 4t^2.$$

The function L acts on the function $y(t) = t^3$ and the result is the function $L(t^3) = 4t^2$. $\lhd$

The function L above is called *an operator*, to emphasize that L is a function that acts on other functions, instead of acting on numbers, as the functions we are used to. The operator L above is also called a *differential operator*, since $L(y)$ contains derivatives of y. These operators are useful to write differential equations in a compact notation, since

$$y'' + a_1(t)\, y' + a_0(t)\, y = f(t)$$

can be written using the operator $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ as

$$L(y) = f.$$

An important type of operators are the linear operators.

Definition 2.1.3. *An operator L is a* ***linear operator*** *iff for every pair of functions y_1, y_2 and constants c_1, c_2 holds*

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2). \tag{2.1.4}$$

In this Section we work with linear operators, as the following result shows.

Theorem 2.1.4 (Linear Operator)**.** *The operator $L(y) = y'' + a_1\, y' + a_0\, y$, where a_1, a_0 are continuous functions and y is a twice differentiable function, is a linear operator.*

Proof of Theorem 2.1.4: This is a straightforward calculation:

$$L(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)'' + a_1\,(c_1y_1 + c_2y_2)' + a_0\,(c_1y_1 + c_2y_2).$$

Recall that derivations is a linear operation and then reorder terms in the following way,

$$L(c_1y_1 + c_2y_2) = \big(c_1y_1'' + a_1\,c_1y_1' + a_0\,c_1y_1\big) + \big(c_2y_2'' + a_1\,c_2y_2' + a_0\,c_2y_2\big).$$

Introduce the definition of L back on the right-hand side. We then conclude that

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

This establishes the Theorem. ☐

The linearity of an operator L translates into the superposition property of the solutions to the homogeneous equation $L(y) = 0$.

Theorem 2.1.5 (Superposition)**.** *If L is a linear operator and y_1, y_2 are solutions of the homogeneous equations $L(y_1) = 0$, $L(y_2) = 0$, then for every constants c_1, c_2 holds*

$$L(c_1\,y_1 + c_2\,y_2) = 0.$$

Remark: This result is *not true* for nonhomogeneous equations.

Proof of Theorem 2.1.5: Verify that the function $y = c_1y_1 + c_2y_2$ satisfies $L(y) = 0$ for every constants c_1, c_2, that is,

$$L(y) = L(c_1y_1 + c_2y_2) = c_1\,L(y_1) + c_2\,L(y_2) = c_1\,0 + c_2\,0 = 0.$$

This establishes the Theorem. ☐

We now introduce the notion of linearly dependent and linearly independent functions.

Definition 2.1.6. *Two functions y_1, y_2 are called **linearly dependent** iff they are proportional. Otherwise, the functions are **linearly independent**.*

Remarks:

(a) Two functions y_1, y_2 are proportional iff there is a constant c such that for all t holds

$$y_1(t) = c\,y_2(t).$$

(b) The function $y_1 = 0$ is proportional to every other function y_2, since holds $y_1 = 0 = 0\,y_2$.

The definitions of linearly dependent or independent functions found in the literature are equivalent to the definition given here, but they are worded in a slight different way. Often in the literature, two functions are called linearly dependent on the interval I iff there exist constants c_1, c_2, not both zero, such that for all $t \in I$ holds

$$c_1y_1(t) + c_2y_2(t) = 0.$$

Two functions are called linearly independent on the interval I iff they are not linearly dependent, that is, the only constants c_1 and c_2 that for all $t \in I$ satisfy the equation

$$c_1y_1(t) + c_2y_2(t) = 0$$

are the constants $c_1 = c_2 = 0$. This wording makes it simple to generalize these definitions to an arbitrary number of functions.

Example 2.1.7.

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are linearly dependent.
(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are linearly independent.

Solution:
Part (a): This is trivial, since $2y_1(t) - y_2(t) = 0$.

Part (b): Find constants c_1, c_2 such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2}\, c_2 = 0, \qquad c_1 + \frac{3\pi}{2}\, c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions y_1 and y_2 are linearly independent. ◁

We now introduce the second main result in this section. If you know two linearly independent solutions to a second order linear homogeneous differential equation, then you actually know all possible solutions to that equation. Any other solution is just a linear combination of the previous two solutions. We repeat that the equation must be homogeneous. This is the closer we can get to a general formula for solutions to second order linear homogeneous differential equations.

Theorem 2.1.7 (General Solution)**.** *If y_1 and y_2 are linearly independent solutions of the equation $L(y) = 0$ on an interval $I \subset \mathbb{R}$, where $L(y) = y'' + a_1\, y' + a_0\, y$, and a_1, a_2 are continuous functions on I, then there are unique constants c_1, c_2 such that every solution y of the differential equation $L(y) = 0$ on I can be written as a linear combination*

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t).$$

Before we prove Theorem 2.1.7, it is convenient to state the following the definitions, which come out naturally from this Theorem.
Definition 2.1.8.

*(a) The functions y_1 and y_2 are **fundamental solutions** of the equation $L(y) = 0$ iff y_1, y_2 are linearly independent and*

$$L(y_1) = 0, \qquad L(y_2) = 0.$$

*(b) The **general solution** of the homogeneous equation $L(y) = 0$ is a two-parameter family of functions $y_{\rm gen}$ given by*

$$y_{\rm gen}(t) = c_1\, y_1(t) + c_2\, y_2(t),$$

where the arbitrary constants c_1, c_2 are the parameters of the family, and y_1, y_2 are fundamental solutions of $L(y) = 0$.

Example 2.1.8. Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

Solution: We first show that y_1 and y_2 are solutions to the differential equation, since

$$L(y_1) = y_1'' + y_1' - 2y_1 = e^t + e^t - 2e^t = (1 + 1 - 2)e^t = 0,$$

$$L(y_2) = y_2'' + y_2' - 2y_2 = 4\, e^{-2t} - 2\, e^{-2t} - 2e^{-2t} = (4 - 2 - 2)e^{-2t} = 0.$$

It is not difficult to see that y_1 and y_2 are linearly independent. It is clear that they are not proportional to each other. A proof of that statement is the following: Find the constants c_1 and c_2 such that

$$0 = c_1\, y_1 + c_2\, y_2 = c_1\, e^t + c_2\, e^{-2t} \quad t \in \mathbb{R} \quad \Rightarrow \quad 0 = c_1\, e^t - 2c_2\, e^{-2t}$$

The second equation is the derivative of the first one. Take $t = 0$ in both equations,

$$0 = c_1 + c_2, \quad 0 = c_1 - 2c_2 \quad \Rightarrow \quad c_1 = c_2 = 0.$$

We conclude that y_1 and y_2 are fundamental solutions to the differential equation above.$\lhd$

Remark: The fundamental solutions to the equation above are not unique. For example, show that another set of fundamental solutions to the equation above is given by,

$$y_1(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}, \qquad y_2(t) = \frac{1}{3}\big(e^t - e^{-2t}\big).$$

To prove Theorem 2.1.7 we need to introduce the Wronskian function and to verify some of its properties. The Wronskian function is studied in the following Subsection and Abel's Theorem is proved. Once that is done we can say that the proof of Theorem 2.1.7 is complete.

Proof of Theorem 2.1.7: We need to show that, given any fundamental solution pair, y_1, y_2, any other solution y to the homogeneous equation $L(y) = 0$ must be a unique linear combination of the fundamental solutions,

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t), \tag{2.1.5}$$

for appropriately chosen constants c_1, c_2.

First, the superposition property implies that the function y above is solution of the homogeneous equation $L(y) = 0$ for every pair of constants c_1, c_2.

Second, given a function y, if there exist constants c_1, c_2 such that Eq. (2.1.5) holds, then these constants are unique. The reason is that functions y_1, y_2 are linearly independent. This can be seen from the following argument. If there are another constants $\tilde{c}_1$, $\tilde{c}_2$ so that

$$y(t) = \tilde{c}_1\, y_1(t) + \tilde{c}_2\, y_2(t),$$

then subtract the expression above from Eq. (2.1.5),

$$0 = (c_1 - \tilde{c}_1)\, y_1 + (c_2 - \tilde{c}_2)\, y_2 \quad \Rightarrow \quad c_1 - \tilde{c}_1 = 0, \quad c_2 - \tilde{c}_2 = 0,$$

where we used that y_1, y_2 are linearly independent. This second part of the proof can be obtained from the part three below, but I think it is better to highlight it here.

So we only need to show that the expression in Eq. (2.1.5) contains all solutions. We need to show that we are not missing any other solution. In this third part of the argument enters Theorem 2.1.2. This Theorem says that, in the case of homogeneous equations, the initial value problem

$$L(y) = 0, \qquad y(t_0) = d_1, \quad y'(t_0) = d_2,$$

always has a unique solution. That means, a good parametrization of all solutions to the differential equation $L(y) = 0$ is given by the two constants, d_1, d_2 in the initial condition. To finish the proof of Theorem 2.1.7 we need to show that the constants c_1 and c_2 are also good to parametrize all solutions to the equation $L(y) = 0$. One way to show this, is to find an invertible map from the constants d_1, d_2, which we know parametrize all solutions, to the constants c_1, c_2. The map itself is simple to find,

$$\begin{aligned} d_1 &= c_1\, y_1(t_0) + c_2\, y_2(t_0) \\ d_2 &= c_1\, y_1'(t_0) + c_2\, y_2'(t_0). \end{aligned}$$

We now need to show that this map is invertible. From linear algebra we know that this map acting on c_1, c_2 is invertible iff the determinant of the coefficient matrix is nonzero,

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0.$$

This leads us to investigate the function

$$W_{12}(t) = y_1(t)\, y_2'(t) - y_1'(t) y_2(t)$$

This function is called the Wronskian of the two functions y_1, y_2. At the end of this section we prove Theorem 2.1.13, which says the following: If y_1, y_2 are fundamental solutions of $L(y) = 0$ on $I \subset \mathbb{R}$, then $W_{12}(t) \neq 0$ on I. This statement establishes the Theorem. $\square$

2.1.4. The Wronskian Function.

We now introduce a function that provides important information about the linear dependency of two functions y_1, y_2. This function, W, is called the Wronskian to honor the polish scientist Josef Wronski, who first introduced this function in 1821 while studying a different problem.

Definition 2.1.9. *The* ***Wronskian*** *of the differentiable functions y_1, y_2 is the function*

$$W_{12}(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t).$$

Remark: Introducing the matrix valued function $A(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$ the Wronskian can be written using the determinant of that 2×2 matrix, $W_{12}(t) = \det\big(A(t)\big)$. An alternative notation is: $W_{12} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

Example 2.1.9. Find the Wronskian of the functions:
(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)
(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

Solution:
Part (a): By the definition of the Wronskian:

$$W_{12}(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix} = \sin(t) 2\cos(t) - \cos(t) 2\sin(t)$$

We conclude that $W_{12}(t) = 0$. Notice that y_1 and y_2 are linearly dependent.

Part (b): Again, by the definition of the Wronskian:

$$W_{12}(t) = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix} = \sin(t)\big[\sin(t) + t\cos(t)\big] - \cos(t) t\sin(t).$$

We conclude that $W_{12}(t) = \sin^2(t)$. Notice that y_1 and y_2 are linearly independent. $\triangleleft$

It is simple to prove the following relation between the Wronskian of two functions and the linear dependency of these two functions.

Theorem 2.1.10 (Wronskian I). *If y_1, y_2 are linearly dependent on $I \subset \mathbb{R}$, then*

$$W_{12} = 0 \quad \text{on} \quad I.$$

Proof of Theorem 2.1.10: Since the functions y_1, y_2 are linearly dependent, there exists a nonzero constant c such that $y_1 = c\, y_2$; hence holds,

$$W_{12} = y_1\, y_2' - y_1'\, y_2 = (c\, y_2)\, y_2' - (c\, y_2)'\, y_2 = 0.$$

This establishes the Theorem. $\square$

Remark: The converse statement to Theorem 2.1.10 is false. If $W_{12}(t) = 0$ for all $t \in I$, that ***does not*** imply that y_1 and y_2 are linearly dependent.

Example 2.1.10. Show that the functions

$$y_1(t) = t^2, \quad \text{and} \quad y_2(t) = |t|\, t, \quad \text{for} \quad t \in \mathbb{R}$$

are linearly independent and have Wronskian $W_{12} = 0$.

Solution:

First, these functions are linearly independent, since $y_1(t) = -y_2(t)$ for $t < 0$, but $y_1(t) = y_2(t)$ for $t > 0$. So there is not c such that $y_1(t) = c\, y_2(t)$ for all $t \in \mathbb{R}$.

Second, their Wronskian vanishes on $\mathbb{R}$. This is simple to see, since $y_1(t) = -y_2(t)$ for $t < 0$, then $W_{12} = 0$ for $t < 0$. Since $y_1(t) = y_2(t)$ for $t > 0$, then $W_{12} = 0$ for $t > 0$. Finally, it is not difficult to see that $W_{12}(t = 0) = 0$. ◁

Remark: Often in the literature one finds the negative of Theorem 2.1.10, which is equivalent to Theorem 2.1.10, and we summarize ibn the followng Corollary.

Corollary 2.1.11 (Wronskian I)**.** *If the Wronskian $W_{12}(t_0) \neq 0$ at a point $t_0 \in I$, then the functions y_1, y_2 defined on I are linearly independent.*

The results mentioned above provide different properties of the Wronskian of two functions. But none of these results is what we need to finish the proof of Theorem 2.1.7. In order to finish that proof we need one more result, Abel's Theorem.

2.1.5. Abel's Theorem.

We now show that the Wronskian of two solutions of a differential equation satisfies a differential equation of its own. This result is known as Abel's Theorem.

Theorem 2.1.12 (Abel)**.** *If y_1, y_2 are twice continuously differentiable solutions of*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{2.1.6}$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then the Wronskian W_{12} satisfies

$$W_{12}' + a_1(t)\, W_{12} = 0.$$

Therefore, for any $t_0 \in I$, the Wronskian W_{12} is given by the expression

$$W_{12}(t) = W_{12}(t_0)\, e^{-A_1(t)},$$

where $A_1(t) = \displaystyle\int_{t_0}^{t} a_1(s)\, ds$.

Proof of Theorem 2.1.12: We start computing the derivative of the Wronskian function,

$$W_{12}' = \big(y_1\, y_2' - y_1'\, y_2\big)' = y_1\, y_2'' - y_1''\, y_2.$$

Recall that both y_1 and y_2 are solutions to Eq. (2.1.6), meaning,

$$y_1'' = -a_1\, y_1' - a_0\, y_1, \qquad y_2'' = -a_1\, y_2' - a_0\, y_2.$$

Replace these expressions in the formula for W_{12}' above,

$$W_{12}' = y_1\, \big(-a_1\, y_2' - a_0\, y_2\big) - \big(-a_1\, y_1' - a_0\, y_1\big)\, y_2 \quad \Rightarrow \quad W_{12}' = -a_1\big(y_1\, y_2' - y_1'\, y_2\big)$$

So we obtain the equation

$$W_{12}' + a_1(t)\; W_{12} = 0.$$

This equation for W_{12} is a first order linear equation; its solution can be found using the method of integrating factors, given in Section 1.1, which results is the expression in the Theorem 2.1.12. This establishes the Theorem. □

We now show one application of Abel's Theorem.

Example 2.1.11. Find the Wronskian of two solutions of the equation

$$t^2\,y'' - t(t+2)\,y' + (t+2)\,y = 0, \qquad t > 0.$$

Solution: Notice that we do not known the explicit expression for the solutions. Nevertheless, Theorem 2.1.12 says that we can compute their Wronskian. First, we have to rewrite the differential equation in the form given in that Theorem, namely,

$$y'' - \Big(\frac{2}{t} + 1\Big)\,y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\,y = 0.$$

Then, Theorem 2.1.12 says that the Wronskian satisfies the differential equation

$$W_{12}'(t) - \Big(\frac{2}{t} + 1\Big)\,W_{12}(t) = 0.$$

This is a first order, linear equation for W_{12}, so its solution can be computed using the method of integrating factors. That is, first compute the integral

$$\begin{aligned} -\int_{t_0}^{t} \Big(\frac{2}{s} + 1\Big)\,ds &= -2\ln\Big(\frac{t}{t_0}\Big) - (t - t_0) \\ &= \ln\Big(\frac{t_0^2}{t^2}\Big) - (t - t_0). \end{aligned}$$

Then, the integrating factor μ is given by

$$\mu(t) = \frac{t_0^2}{t^2}\,e^{-(t-t_0)},$$

which satisfies the condition $\mu(t_0) = 1$. So the solution, W_{12} is given by

$$\big(\mu(t)W_{12}(t)\big)' = 0 \quad \Rightarrow \quad \mu(t)W_{12}(t) - \mu(t_0)W_{12}(t_0) = 0$$

so, the solution is

$$W_{12}(t) = W_{12}(t_0)\,\frac{t^2}{t_0^2}\,e^{(t-t_0)}.$$

If we call the constant $c = W_{12}(t_0)/[t_0^2 e^{t_0}]$, then the Wronskian has the simpler form

$$W_{12}(t) = c\,t^2 e^t.$$

$\lhd$

We now state and prove the statement we need to complete the proof of Theorem 2.1.7.

Theorem 2.1.13 (Wronskian II). *If y_1, y_2 are fundamental solutions of $L(y) = 0$ on $I \subset \mathbb{R}$, then $W_{12}(t) \neq 0$ on I.*

Remark: Instead of proving the Theorem above, we prove an equivalent statement—the negative statement.

Corollary 2.1.14 (Wronskian II). *If y_1, y_2 are solutions of $L(y) = 0$ on $I \subset \mathbb{R}$ and there is a point $t_1 \in I$ such that $W_{12}(t_1) = 0$, then y_1, y_2 are linearly dependent on I.*

Proof of Corollary 2.1.14: We know that y_1, y_2 are solutions of $L(y) = 0$. Then, Abel's Theorem says that their Wronskian W_{12} is given by

$$W_{12}(t) = W_{12}(t_0)\,e^{-A_1(t)},$$

for any $t_0 \in I$. Chossing the point t_0 to be t_1, the point where by hypothesis $W_{12}(t_1) = 0$, we get that

$$W_{12}(t) = 0 \quad \text{for all} \quad t \in I.$$

Knowing that the Wronskian vanishes identically on I, we can write

$$y_1\, y_2' - y_1'\, y_2 = 0,$$

on I. If either y_1 or y_2 is the function zero, then the set is linearly dependent. So we can assume that both are not identically zero. Let's assume there exists $t_1 \in I$ such that $y_1(t_1) \neq 0$. By continuity, y_1 is nonzero in an open neighborhood $I_1 \subset I$ of t_1. So in that neighborhood we can divide the equation above by y_1^2,

$$\frac{y_1\, y_2' - y_1'\, y_2}{y_1^2} = 0 \quad \Rightarrow \quad \Big(\frac{y_2}{y_1}\Big)' = 0 \quad \Rightarrow \quad \frac{y_2}{y_1} = c, \qquad \text{on } \ I_1,$$

where $c \in \mathbb{R}$ is an arbitrary constant. So we conclude that y_1 is proportional to y_2 on the open set I_1. That means that the function $y(t) = y_2(t) - c\, y_1(t)$, satisfies

$$L(y) = 0, \qquad y(t_1) = 0, \quad y'(t_1) = 0.$$

Therefore, the existence and uniqueness Theorem 2.1.2 says that $y(t) = 0$ for all $t \in I$. This finally shows that y_1 and y_2 are linearly dependent. This establishes the Theorem. ☐

2.1.6. Exercises.

2.1.1.- Find the constants c and k such that the function $y(t) = c\,t^k$ is solution of

$$-t^3\,y + t^2\,y + 4t\,y = 1.$$

2.1.2.- Let $y(t) = c_1\,t + c_2\,t^2$ be the general solution of a second order linear differential equation $L(y) = 0$. By eliminating the constants c_1 and c_2, find the differential equation satisfied by y.

2.1.3.- (a) Verify that $y_1(t) = t^2$ and $y_2(t) = 1/t$ are solutions to the differential equation

$$t^2 y'' - 2y = 0, \quad t > 0.$$

(b) Show that $y(t) = a\,t^2 + \dfrac{b}{t}$ is solution of the same equation for all constants $a, b \in \mathbb{R}$.

2.1.4.- Find the longest interval where the solution y of the initial value problems below is defined. (Do not try to solve the differential equations.)

(a) $t^2 y'' + 6y = 2t$, $y(1) = 2$, $y'(1) = 3$.

(b) $(t-6)y' + 3ty' - y = 1$, $y(3) = -1$, $y'(3) = 2$.

2.1.5.- If the graph of y, solution to a second order linear differential equation $L(y(t)) = 0$ on the interval $[a, b]$, is tangent to the t-axis at any point $t_0 \in [a, b]$, then find the solution y explicitly.

2.1.6.- Can the function $y(t) = \sin(t^2)$ be solution on an open interval containing $t = 0$ of a differential equation

$$y'' + a(t)\,y' + b(t)y = 0,$$

with continuous coefficients a and b? Explain your answer.

2.1.7.- Compute the Wronskian of the following functions:

(a) $f(t) = \sin(t)$, $g(t) = \cos(t)$.

(b) $f(x) = x$, $g(x) = x\,e^x$.

(c) $f(\theta) = \cos^2(\theta)$, $g(\theta) = 1 + \cos(2\theta)$.

2.1.8.- Verify whether the functions y_1, y_2 below are a fundamental set for the differential equations given below:

(a) $y_1(t) = \cos(2t)$, $y_2(t) = \sin(2t)$,

$$y'' + 4y = 0.$$

(b) $y_1(t) = e^t$, $y_2(t) = t\,e^t$,

$$y'' - 2y' + y = 0.$$

(c) $y_1(x) = x$, $y_2(t) = x\,e^x$,

$$x^2\,y'' - 2x(x+2)\,y' + (x+2)\,y = 0.$$

2.1.9.- If the Wronskian of any two solutions of the differential equation

$$y'' + p(t)\,y' + q(t)\,y = 0$$

is constant, what does this imply about the coefficients p and q?

2.1.10.- * Suppose y_1 is solution of the IVP

$$y_1'' + a_1\,y_1' + a_0\,y_1 = 0, \qquad \begin{aligned} y_1(0) &= 0,\\ y_1'(0) &= 5, \end{aligned}$$

and y_2 is solution of the IVP

$$y_1'' + a_1\,y_1' + a_0\,y_1 = 0, \qquad \begin{aligned} y_1(0) &= 0,\\ y_1'(0) &= 1 \end{aligned}$$

that is, same differential equation and same initial condition for the function, but different initial conditions for the derivatives. Then show that the functions y_1 and y_2 must be proportional to each other,

$$y_1(t) = c\,y_2(t)$$

and find the proportionality factor c.

Hint 1: Theorem 2.1.2 says that the initial value problem

$$y'' + a_1\,y' + a_0\,y = 0, \qquad \begin{aligned} y(0) &= 0,\\ y'(0) &= 0, \end{aligned}$$

has a unique solution and it is $y(t) = 0$ for all t.

Hint 2: Find what is the initial value problem for the function

$$y_c(t) = y_1(t) - c\,y_2(t),$$

and fine tune c to use hint 1.

2.2. Reduction of Order Methods

Sometimes a solution to a second order differential equation can be obtained solving two first order equations, one after the other. When that happens we say we have reduced the order of the equation. We use the ideas in Chapter 1 to solve each first order equation.

In this section we focus on three types of differential equations where such reduction of order happens. The first two cases are usually called special second order equations and the third case is called the conservation of the energy.

We end this section with a method that provides a second solution to a second order equation if you already know one solution. The second solution can be chosen not proportional to the first one. This idea is called the reduction order method—although all four ideas we study in this section do reduce the order of the original equation.

2.2.1. Special Second Order Equations.

A second order differential equation is called special when either the function, or its first derivative, or the independent variable does not appear explicitly in the equation. In these cases the second order equation can be transformed into a first order equation for a new function. The transformation to get the new function is different in each case. Then, one solves the first order equation and transforms back solving another first order equation to get the original function. We start with a few definitions.

Definition 2.2.1. *A* ***second order*** *equation in the unknown function y is an equation*

$$y'' = f(t, y, y').$$

where the function $f : \mathbb{R}^3 \to \mathbb{R}$ is given. The equation is ***linear*** *iff function f is linear in both arguments y and y'. The second order differential equation above is* ***special*** *iff one of the following conditions hold:*

(a) $y'' = f(t, \cancel{y}, y')$, *the function y does not appear explicitly in the equation;*
(b) $y'' = f(\cancel{t}, y, y')$, *the variable t does not appear explicitly in the equation.*
(c) $y'' = f(\cancel{t}, y, \cancel{y'})$, *the variable t, the function y' do not appear explicitly in the equation.*

It is simpler to solve special second order equations when the function y is missing, case *(a)*, than when the variable t is missing, case *(b)*, as it can be seen by comparing Theorems 2.2.2 and 2.2.3. The case *(c)* is well known in physics, since it applies to Newton's second law of motion in the case that the force on a particle depends only on the position of the particle. In such a case one can show that the *energy of the particle is conserved.*

Let us start with case *(a)*.

Theorem 2.2.2 (Function y Missing)**.** *If a second order differential equation has the form $y'' = f(t, y')$, then $v = y'$ satisfies the first order equation $v' = f(t, v)$.*

The proof is trivial, so we go directly to an example.

Example 2.2.1. Find the y solution of the second order nonlinear equation $y'' = -2t\,(y')^2$ with initial conditions $y(0) = 2$, $y'(0) = -1$.

Solution: Introduce $v = y'$. Then $v' = y''$, and

$$v' = -2t\,v^2 \quad \Rightarrow \quad \frac{v'}{v^2} = -2t \quad \Rightarrow \quad -\frac{1}{v} = -t^2 + c.$$

So, $\dfrac{1}{y'} = t^2 - c$, that is, $y' = \dfrac{1}{t^2 - c}$. The initial condition implies

$$-1 = y'(0) = -\frac{1}{c} \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y' = \frac{1}{t^2 - 1}.$$

Then, $y = \displaystyle\int \frac{dt}{t^2 - 1} + c$. We integrate using the method of partial fractions,

$$\frac{1}{t^2 - 1} = \frac{1}{(t-1)(t+1)} = \frac{a}{(t-1)} + \frac{b}{(t+1)}.$$

Hence, $1 = a(t+1) + b(t-1)$. Evaluating at $t = 1$ and $t = -1$ we get $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$. So

$$\frac{1}{t^2 - 1} = \frac{1}{2}\Big[\frac{1}{(t-1)} - \frac{1}{(t+1)}\Big].$$

Therefore, the integral is simple to do,

$$y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + c. \qquad 2 = y(0) = \frac{1}{2}(0 - 0) + c.$$

We conclude $y = \dfrac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + 2$. ◁

The case *(b)* is way more complicated to solve.

Theorem 2.2.3 (Variable t Missing)**.** *If the initial value problem*

$$y'' = f(y, y'), \qquad y(0) = y_0, \qquad y'(0) = y_1,$$

has an invertible solution y, then the function

$$w(y) = v(t(y)),$$

where $v(t) = y'(t)$, and $t(y)$ is the inverse of $y(t)$, satisfies the initial value problem

$$\dot{w} = \frac{f(y, w)}{w}, \qquad w(y_0) = y_1,$$

where we denoted $\dot{w} = \dfrac{dw}{dy}$.

Remark: The proof is based on the chain rule for the derivative of functions.

Proof of Theorem 2.2.3: The differential equation is $y'' = f(y, y')$. Denoting $v(t) = y'(t)$

$$v' = f(y, v)$$

It is not clear how to solve this equation, since the function y still appears in the equation. On a domain where y is invertible we can do the following. Denote $t(y)$ the inverse values of $y(t)$, and introduce $w(y) = v(t(y))$. The chain rule implies

$$\dot{w}(y) = \frac{dw}{dy}\Big|_{y} = \frac{dv}{dt}\Big|_{t(y)} \frac{dt}{dy}\Big|_{t(y)} = \frac{v'(t)}{y'(t)}\Big|_{t(y)} = \frac{v'(t)}{v(t)}\Big|_{t(y)} = \frac{f\big(y(t), v(t)\big)}{v(t)}\Big|_{t(y)} = \frac{f(y, w(y))}{w(y)}.$$

where $\dot{w}(y) = \dfrac{dw}{dy}$, and $v'(t) = \dfrac{dv}{dt}$. Therefore, we have obtained the equation for w, namely

$$\dot{w} = \frac{f(y, w)}{w}$$

Finally we need to find the initial condition fro w. Recall that

$$\begin{aligned} y(t=0) &= y_0 &\Leftrightarrow&& t(y=y_0) &= 0,\\ y'(t=0) &= y_1 &\Leftrightarrow&& v(t=0) &= y_1. \end{aligned}$$

Therefore,

$$w(y=y_0) = v(t(y=y_0)) = v(t=0) = y_1 \quad \Rightarrow \quad w(y_0) = y_1.$$

This establishes the Theorem. □

Example 2.2.2. Find a solution y to the second order equation $y'' = 2y\,y'$.

Solution: The variable t does not appear in the equation. So we start introduciong the function $v(t) = y'(t)$. The equation is now given by $v'(t) = 2y(t)\,v(t)$. We look for invertible solutions y, then introduce the function $w(y) = v(t(y))$. This function satisfies

$$\dot{w}(y) = \frac{dw}{dy} = \Big(\frac{dv}{dt}\,\frac{dt}{dy}\Big)\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)}.$$

Using the differential equation,

$$\dot{w}(y) = \frac{2yv}{v}\Big|_{t(y)} \quad \Rightarrow \quad \frac{dw}{dy} = 2y \quad \Rightarrow \quad w(y) = y^2 + c.$$

Since $v(t) = w(y(t))$, we get $v(t) = y^2(t) + c$. This is a separable equation,

$$\frac{y'(t)}{y^2(t)+c} = 1.$$

Since we only need to find a solution of the equation, and the integral depends on whether $c > 0$, $c = 0$, $c < 0$, we choose (for no special reason) only one case, $c = 1$.

$$\int \frac{dy}{1+y^2} = \int dt + c_0 \quad \Rightarrow \quad \arctan(y) = t + c_0 y(t) = \tan(t + c_0).$$

Again, for no reason, we choose $c_0 = 0$, and we conclude that one possible solution to our problem is $y(t) = \tan(t)$. ◁

Example 2.2.3. Find the solution y to the initial value problem

$$y\,y'' + 3(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 6.$$

Solution: We start rewriting the equation in the standard form

$$y'' = -3\,\frac{(y')^2}{y}.$$

The variable t does not appear explicitly in the equation, so we introduce the function $v(t) = y'(t)$. The differential equation now has the form $v'(t) = -3v^2(t)/y(t)$. We look for invertible solutions y, and then we introduce the function $w(y) = v(t(y))$. Because of the chain rule for derivatives, this function satisfies

$$\dot{w}(y) = \frac{dw}{dy}(y) = \Big(\frac{dv}{dt}\,\frac{dt}{dy}\Big)\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)} \quad \Rightarrow \quad \dot{w}(y) = \frac{v'(t(y))}{w(y)}.$$

Using the differential equation on the factor v', we get

$$\dot{w}(y) = \frac{-3v^2(t(y))}{y}\,\frac{1}{w} = \frac{-3w^2}{yw} \quad \Rightarrow \quad \dot{w} = \frac{-3w}{y}.$$

This is a separable equation for function w. The problem for w also has initial conditions, which can be obtained from the initial conditions from y. Recalling the definition of inverse function,

$$y(t=0)=1 \quad \Leftrightarrow \quad t(y=1)=0.$$

Therefore,

$$w(y=1)=v(t(y=1))=v(0)=y'(0)=6,$$

where in the last step above we use the initial condition $y'(0)=6$. Summarizing, the initial value problem for w is

$$\dot{w}=\frac{-3w}{y}, \quad w(1)=6.$$

The equation for w is separable, so the method from § 1.3 implies that

$$\ln(w)=-3\ln(y)+c_0=\ln(y^{-3})+c_0 \quad \Rightarrow \quad w(y)=c_1\,y^{-3}, \quad c_1=e^{c_0}.$$

The initial condition fixes the constant c_1, since

$$6=w(1)=c_1 \quad \Rightarrow \quad w(y)=6\,y^{-3}.$$

We now transform from w back to v as follows,

$$v(t)=w(y(t))=6\,y^{-3}(t) \quad \Rightarrow \quad y'(t)=6y^{-3}(t).$$

This is now a first order separable equation for y. Again the method from § 1.3 imply that

$$y^3\,y'=6 \quad \Rightarrow \quad \frac{y^4}{4}=6t+c_2$$

The initial condition for y fixes the constant c_2, since

$$1=y(0) \quad \Rightarrow \quad \frac{1}{4}=0+c_2 \quad \Rightarrow \quad \frac{y^4}{4}=6t+\frac{1}{4}.$$

So we conclude that the solution y to the initial value problem is

$$y(t)=(24t+1)^4.$$

◁

2.2.2. Conservation of the Energy. We now study case *(c)* in Def. 2.2.1—second order differential equations such that both the variable t and the function y' do not appear explicitly in the equation. This case is important in Newtonian mechanics. For that reason we slightly change notation we use to write the differential equation. Instead of writing the equation as $y''=f(y)$, as in Def. 2.2.1, we write it as

$$m\,y''=f(y),$$

where m is a constant. This notation matches the notation of Newton's second law of motion for a particle of mass m, with position function y as function of time t, acting under a force f that depends only on the particle position y.

It turns out that solutions to the differential equation above have a particular property: There is a function of y' and y, called the energy of the system, that remains conserved during the motion. We summarize this result in the statement below.

Theorem 2.2.4 (Conservation of the Energy)**.** *Consider a particle with positive mass m and position y, function of time t, which is a solution of Newton's second law of motion*

$$m\,y''=f(y),$$

with initial conditions $y(t_0) = y_0$ *and* $y'(t_0) = v_0$*, where* $f(y)$ *is the force acting on the particle at the position* y*. Then, the position function* y *satisfies*

$$\frac{1}{2}mv^2 + V(y) = E_0,$$

where $E_0 = \frac{1}{2}mv_0^2 + V(y_0)$ *is fixed by the initial conditions,* $v(t) = y'(t)$ *is the particle velocity, and* V *is the potential of the force* f*—the negative of the primitive of* f*, that is*

$$V(y) = -\int f(y)\,dy \quad \Leftrightarrow \quad f = -\frac{dV}{dy}.$$

Remark: The term $T(v) = \frac{1}{2}mv^2$ is the kinetic energy of the particle. The term $V(y)$ is the potential energy. The Theorem above says that the total mechanical energy

$$E = T(v) + V(y)$$

remains constant during the motion of the particle.

Proof of Theorem 2.2.4: We write the differential equation using the potential V,

$$m\,y'' = -\frac{dV}{dy}.$$

Multiply the equation above by y',

$$m\,y'(t)\,y''(t) = -\frac{dV}{dy}\,y'(t).$$

Use the chain rule on both sides of the equation above,

$$\frac{d}{dt}\Big(\frac{1}{2}m\,(y')^2\Big) = -\frac{d}{dt}V(y(t)).$$

Introduce the velocity $v = y'$, and rewrite the equation above as

$$\frac{d}{dt}\Big(\frac{1}{2}m\,v^2 + V(y)\Big) = 0.$$

This means that the quantity

$$E(y,v) = \frac{1}{2}m\,v^2 + V(y),$$

called the mechanical energy of the system, remains constant during the motion. Therefore, it must match its value at the initial time t_0, which we called E_0 in the Theorem. So we arrive to the equation

$$E(y,v) = \frac{1}{2}m\,v^2 + V(y) = E_0.$$

This establishes the Theorem. □

Example 2.2.4. Find the potential energy and write the energy conservation for the following systems:

(i) A particle attached to a spring with constant k, moving in one space dimension.
(ii) A particle moving vertically close to the Earth surface, under Earth's constant gravitational acceleration. In this case the force on the particle having mass m is $f(y) = mg$, where $g = 9.81\ \text{m/s}^2$.
(iii) A particle moving along the direction vertical to the surface of a spherical planet with mass M and radius R.

Solution:

Case *(i)*. The force on a particle of mass m attached to a spring with spring constant $k > 0$, when displaced an amount y from the equilibrium position $y = 0$ is $f(y) = -ky$. Therefore, Newton's second law of motion says

$$m\,y'' = -ky.$$

The potential in this case is $V(y) = \frac{1}{2}ky^2$, since $-dV/dy = -ky = f$. If we introduce the particle velocity $v = y'$, then the total mechanical energy is

$$E(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2.$$

The conservation of the energy says that

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0,$$

where E_0 is the energy at the initial time.

Case *(ii)*. Newton's equation of motion says: $m\,y'' = m\,g$. If we multiply Newton's equation by y', we get

$$m\,y'\,y'' = m\,g\,y' \quad \Rightarrow \quad \frac{d}{dt}\Big(\frac{1}{2}\,m\,(y')^2 + m\,g\,y\Big) = 0$$

If we introduce the the gravitational energy

$$E(y, v) = \frac{1}{2}mv^2 + mgy,$$

where $v = y'$, then Newton's law says $\dfrac{dE}{dt} = 0$, so the total gravitational energy is constant,

$$\frac{1}{2}mv^2 + mgy = E(0).$$

Case *(iii)*. Consider a particle of mass m moving on a line which is perpendicular to the surface of a spherical planet of mass M and radius R. The force on such a particle when is at a distance y from the surface of the planet is, according to Newton's gravitational law,

$$f(y) = -\frac{GMm}{(R + y)^2},$$

where $G = 6.67 \times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{s}^2\,\mathrm{Kg}}$, is Newton's gravitational constant. The potential is

$$V(y) = -\frac{GMm}{(R + y)},$$

since $-dV/dy = f(y)$. The energy for this system is

$$E(y, v) = \frac{1}{2}mv^2 - \frac{GMm}{(R + y)}$$

where we introduced the particle velocity $v = y'$. The conservation of the energy says that

$$\frac{1}{2}mv^2 - \frac{GMm}{(R + y)} = E_0,$$

where E_0 is the energy at the initial time. ◁

Example 2.2.5. Find the maximum height of a ball of mass $m = 0.1$ Kg that is shot vertically by a spring with spring constant $k = 400$ Kg/s^2 and compressed 0.1 m. Use $g = 10$ m/s^2.

Solution: This is a difficult problem to solve if one tries to find the position function y and evaluate it at the time when its speed vanishes—maximum altitude. One has to solve two differential equations for y, one with source $f_1 = -ky - mg$ and other with source $f_2 = -mg$, and the solutions must be glued together. The first source describes the particle when is pushed by the spring under the Earth's gravitational force. The second source describes the particle when only the Earth's gravitational force is acting on the particle. Also, the moment when the ball leaves the spring is hard to describe accurately.

A simpler method is to use the conservation of the mechanical and gravitational energy. The energy for this particle is

$$E(t) = \frac{1}{2}\,mv^2 + \frac{1}{2}\,ky^2 + mgy.$$

This energy must be constant along the movement. In particular, the energy at the initial time $t = 0$ must be the same as the energy at the time of the maximum height, t_M,

$$E(t=0) = E(t_M) \quad \Rightarrow \quad \frac{1}{2}\,mv_0^2 + \frac{1}{2}\,ky_0^2 + mgy_0 = \frac{1}{2}\,mv_M^2 + mgy_M.$$

But at the initial time we have $v_0 = 0$, and $y_0 = -0.1$, (the negative sign is because the spring is compressed) and at the maximum time we also have $v_M = 0$, hence

$$\frac{1}{2}\,ky_0^2 + mgy_0 = mgy_M \quad \Rightarrow \quad y_M = y_0 + \frac{k}{2mg}\,y_0^2.$$

We conclude that $y_M = 1.9$ m. ◁

Example 2.2.6. Find the escape velocity from Earth—the initial velocity of a projectile moving vertically upwards starting from the Earth surface such that it escapes Earth gravitational attraction. Recall that the acceleration of gravity at the surface of Earth is $g = GM/R^2 = 9.81\,\mathrm{m/s^2}$, and that the radius of Earth is $R = 6378\,\mathrm{Km}$. Here M denotes the mass of the Earth, and G is Newton's gravitational constant.

Solution: The projectile moves in the vertical direction, so the movement is along one space dimension. Let y be the position of the projectile, with $y = 0$ at the surface of the Earth. Newton's equation in this case is

$$m\,y'' = -\frac{GMm}{(R+y)^2}.$$

We start rewriting the force using the constant g instead of G,

$$-\frac{GMm}{(R+y)^2} = -\frac{GM}{R^2}\,\frac{mR^2}{(R+y)^2} = -\frac{gmR^2}{(R+y)^2}.$$

So the equation of motion for the projectile is

$$m\,y'' = -\frac{gmR^2}{(R+y)^2}.$$

The projectile mass m can be canceled from the equation above (we do it later) so the result will be independent of the projectile mass. Now we introduce the gravitational potential

$$V(y) = -\frac{gmR^2}{(R+y)}.$$

We know that the motion of this particle satisfies the conservation of the energy

$$\frac{1}{2}mv^2 - \frac{gmR^2}{(R+y)} = E_0,$$

where $v = y'$. The initial energy is simple to compute, $y(0) = 0$ and $v(0) = v_0$, so we get

$$\frac{1}{2}mv^2(t) - \frac{gmR^2}{(R+y(t))} = \frac{1}{2}mv_0^2 - gmR.$$

We now cancel the projectile mass from the equation, and we rewrite the equation as

$$v^2(t) = v_0^2 - 2gR + \frac{2gR^2}{(R+y(t))}.$$

Now we choose the initial velocity v_0 to be the escape velocity v_e. The latter is the smallest initial velocity such that $v(t)$ is defined for all y including $y \to \infty$. Since

$$v^2(t) \geqslant 0 \qquad \text{and} \qquad \frac{2gR^2}{(R+y(t))} > 0,$$

this means that the escape velocity must satisfy

$$v_e^2 - 2gR \geqslant 0.$$

Since the escape velocity is the smallest velocity satisfying the condition above, that means

$$v_e = \sqrt{2gR} \quad \Rightarrow \quad v_e = 11.2\,\text{Km/s}.$$

◁

Example 2.2.7. Find the time t_M for a rocket to reach the Moon, if it is launched at the escape velocity. Use that the distance from the surface of the Earth to the Moon is $d = 405,696$ Km.

Solution: From Example 2.2.6 we know that the position function y of the rocket satisfies the differential equation

$$v^2(t) = v_0^2 - 2gR + \frac{2gR^2}{(R+y(t))},$$

where R is the Earth radius, g the gravitational acceleration at the Earth surface, $v = y'$, and v_0 is the initial velocity. Since the rocket initial velocity is the Earth escape velocity, $v_0 = v_e = \sqrt{2gR}$, the differential equation for y is

$$(y')^2 = \frac{2gR^2}{(R+y)} \quad \Rightarrow \quad y' = \frac{\sqrt{2g}\,R}{\sqrt{R+y}},$$

where we chose the positive square root because, in our coordinate system, the rocket leaving Earth means $v > 0$. Now, the last equation above is a separable differential equation for y, so we can integrate it,

$$(R+y)^{1/2}\,y' = \sqrt{2g}\,R \quad \Rightarrow \quad \frac{2}{3}(R+y)^{3/2} = \sqrt{2g}\,R\,t + c,$$

where c is a constant, which can be determined by the initial condition $y(t=0) = 0$, since at the initial time the projectile is on the surface of the Earth, the origin of out coordinate system. With this initial condition we get

$$c = \frac{2}{3}R^{3/2} \quad \Rightarrow \quad \frac{2}{3}(R+y)^{3/2} = \sqrt{2g}\,R\,t + \frac{2}{3}R^{3/2}. \tag{2.2.1}$$

From the equation above we can compute an explicit form of the solution function y,

$$y(t) = \Big(\frac{3}{2}\sqrt{2g}\,R\,t + R^{3/2}\Big)^{2/3} - R. \tag{2.2.2}$$

To find the time to reach the Moon we need to evaluate Eq. (2.2.1) for $y = d$ and get t_M,

$$\frac{2}{3}(R+d)^{3/2} = \sqrt{2g}\,R\,t_M + \frac{2}{3}R^{3/2}. \quad \Rightarrow \quad t_M = \frac{2}{3}\,\frac{1}{\sqrt{2g}\,R}\,\big((R+d)^{3/2} - R^{2/3}\big).$$

The formula above gives $t_M = 51.5\,\text{hours}$. $\lhd$

2.2.3. The Reduction of Order Method. If we know one solution to a second order, *linear, homogeneous,* differential equation, then one can find a second solution to that equation. And this second solution can be chosen to be not proportional to the known solution. One obtains the second solution transforming the original problem into solving two first order differential equations.

Theorem 2.2.5 (Reduction of Order)**.** *If a nonzero function y_1 is solution to*

$$y'' + a_1(t)\,y' + a_0(t)\,y = 0. \tag{2.2.3}$$

where a_1, a_0 are given functions, then a second solution not proportional to y_1 is

$$y_2(t) = y_1(t) \int \frac{e^{-A_1(t)}}{y_1^2(t)}\,dt, \tag{2.2.4}$$

where $A_1(t) = \int a_1(t)\,dt$.

Remark: In the first part of the proof we write $y_2(t) = v(t)\,y_1(t)$ and show that y_2 is solution of Eq. (2.2.3) iff the function v is solution of

$$v'' + \Big(2\frac{y_1'(t)}{y_1(t)} + a_1(t)\Big)\,v' = 0. \tag{2.2.5}$$

In the second part we solve the equation for v. This is a first order equation for for $w = v'$, since v itself does not appear in the equation, hence the name reduction of order method. The equation for w is linear and first order, so we can solve it using the integrating factor method. One more integration gives v, which is the factor multiplying y_1 in Eq. (2.2.4).

Remark: The functions v and w in this subsection have no relation with the functions v and w from the previous subsection.

Proof of Theorem 2.2.5: We write $y_2 = vy_1$ and we put this function into the differential equation in 2.2.3, which give us an equation for v. To start, compute y_2' and y_2'',

$$y_2' = v'\,y_1 + v\,y_1', \qquad y_2'' = v''\,y_1 + 2v'\,y_1' + v\,y_1''.$$

Introduce these equations into the differential equation,

$$\begin{aligned} 0 &= (v''\,y_1 + 2v'\,y_1' + v\,y_1'') + a_1\,(v'\,y_1 + v\,y_1') + a_0 v\,y_1 \\ &= y_1\,v'' + (2y_1' + a_1\,y_1)\,v' + (y_1'' + a_1\,y_1' + a_0\,y_1)\,v. \end{aligned}$$

The function y_1 is solution to the differential original differential equation,

$$y_1'' + a_1\,y_1' + a_0\,y_1 = 0,$$

then, the equation for v is given by

$$y_1\,v'' + (2y_1' + a_1\,y_1)\,v' = 0. \quad \Rightarrow \quad v'' + \Big(2\frac{y_1'}{y_1} + a_1\Big)\,v' = 0.$$

This is Eq. (2.2.5). The function v does not appear explicitly in this equation, so denoting $w = v'$ we obtain

$$w' + \Big(2\frac{y_1'}{y_1} + a_1\Big)\,w = 0.$$

This is is a first order linear equation for w, so we solve it using the integrating factor method, with integrating factor

$$\mu(t) = y_1^2(t)\, e^{A_1(t)}, \quad \text{where} \quad A_1(t) = \int a_1(t)\, dt.$$

Therefore, the differential equation for w can be rewritten as a total t-derivative as

$$\big(y_1^2\, e^{A_1}\, w\big)' = 0 \quad \Rightarrow \quad y_1^2\, e^{A_1}\, w = w_0 \quad \Rightarrow \quad w(t) = w_0\, \frac{e^{-A_1(t)}}{y_1^2(t)}.$$

Since $v' = w$, we integrate one more time with respect to t to obtain

$$v(t) = w_0 \int \frac{e^{-A_1(t)}}{y_1^2(t)}\, dt + v_0.$$

We are looking for just one function v, so we choose the integration constants $w_0 = 1$ and $v_0 = 0$. We then obtain

$$v(t) = \int \frac{e^{-A_1(t)}}{y_1^2(t)}\, dt \quad \Rightarrow \quad y_2(t) = y_1(t) \int \frac{e^{-A_1(t)}}{y_1^2(t)}\, dt.$$

For the furthermore part, we now need to show that the functions y_1 and $y_2 = vy_1$ are linearly independent. We start computing their Wronskian,

$$W_{12} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v'y_1 + vy_1') \end{vmatrix} = y_1(v'y_1 + vy_1') - vy_1y_1' \quad \Rightarrow \quad W_{12} = v'y_1^2.$$

Recall that above in this proof we have computed $v' = w$, and the result was $w = w_0\, e^{-A_1}/y_1^2$. So we get $v'y_1^2 = w_0\, e^{-A_1}$, and then the Wronskian is given by

$$W_{12} = w_0\, e^{-A_1}.$$

This is a nonzero function, therefore the functions y_1 and $y_2 = vy_1$ are linearly independent. This establishes the Theorem. □

Example 2.2.8. Find a second solution y_2 linearly independent to the solution $y_1(t) = t$ of the differential equation

$$t^2 y'' + 2ty' - 2y = 0.$$

Solution: We look for a solution of the form $y_2(t) = t\, v(t)$. This implies that

$$y_2' = t\, v' + v, \qquad y_2'' = t\, v'' + 2v'.$$

So, the equation for v is given by

$$\begin{aligned} 0 &= t^2\big(t\, v'' + 2v'\big) + 2t\big(t\, v' + v\big) - 2t\, v \\ &= t^3\, v'' + (2t^2 + 2t^2)\, v' + (2t - 2t)\, v \\ &= t^3\, v'' + (4t^2)\, v' \quad \Rightarrow \quad v'' + \frac{4}{t} v' = 0. \end{aligned}$$

Notice that this last equation is precisely Eq. (**??**), since in our case we have

$$y_1 = t, \qquad p(t) = \frac{2}{t} \quad \Rightarrow \quad t\, v'' + \big[2 + \frac{2}{t} t\big]\, v' = 0.$$

The equation for v is a first order equation for $w = v'$, given by

$$\frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \qquad c_1 \in \mathbb{R}.$$

Therefore, integrating once again we obtain that

$$v = c_2 t^{-3} + c_3, \qquad c_2, c_3 \in \mathbb{R},$$

and recalling that $y_2 = t\,v$ we then conclude that

$$y_2 = c_2 t^{-2} + c_3 t.$$

Choosing $c_2 = 1$ and $c_3 = 0$ we obtain that $y_2(t) = t^{-2}$. Therefore, a fundamental solution set to the original differential equation is given by

$$y_1(t) = t, \qquad y_2(t) = \frac{1}{t^2}.$$

$\triangleleft$

2.2.4. Exercises.

2.2.1.- Consider the differential equation

$$t^2\,y'' + 3t\,y' - 3 = 0, \quad t > 0,$$

with initial conditions

$$y(1) = 3, \quad y'(1) = \frac{3}{2}.$$

(a) Find the initial value problem satisfied by $v(t) = y'(t)$.
(b) Solve the differential equation for v.
(c) Find the solution y of the differential equation above.

2.2.2.- Consider the differential equation

$$y\,y'' + 3\,(y')^2 = 0,$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 5.$$

(a) Find the differential equation satisfied by $w(y) = v(t(y))$, where $v(t) = y'(t)$.
(b) Find the initial condition satisfied by the function w.
(c) Solve the initial value problem for the function w.
(d) Use the solution w found above to set up a first order initial value problem for y.
(e) Find the solution y of the differential equation above.

2.2.3.- Solve the differential equation

$$y'' = -\frac{y'}{y^7},$$

with initial conditions

$$y(0) = 1, \quad y'(0) = \frac{1}{6}.$$

2.2.4.- Use the reduction order method to find a second solution y_2 to the differential equation

$$t^2\,y'' + 8t\,y' + 12\,y = 0,$$

knowing that $y_1(t) = 1/t^3$ is a solution. The second solution y_2 must not contain any term proportional to y_1.

2.2.5.- * Use the reduction order method to find a solution y_2 of the equation

$$t^2\,y'' + 2t\,y' - 6\,y = 0$$

knowing that $y_1 = t^2$ is a solution to this equation.

(a) Write $y_2 = v\,y_1$ and find y_2' and y_2''.
(b) Find the differential equation for v.
(c) Solve the differential equation for v and find the general solution.
(d) Choose the simplest function v such that y_2 and y_1 are fundamental solutions of the differential equation above.

2.3. Homogenous Constant Coefficients Equations

All solutions to a second order linear homogeneous equation can be obtained from any pair of nonproportional solutions. This is the main idea given in § 2.1, Theorem 2.1.7. In this section we obtain these two linearly independent solutions in the particular case that the equation has constant coefficients. Such problem reduces to solve for the roots of a degree-two polynomial, the characteristic polynomial.

2.3.1. The Roots of the Characteristic Polynomial.

Thanks to the work done in § 2.1 we only need to find two linearly independent solutions to second order linear homogeneous equations. Then Theorem 2.1.7 says that every other solution is a linear combination of the former two. How do we find any pair of linearly independent solutions? Since the equation is so simple, having constant coefficients, we find such solutions by trial and error. Here is an example of this idea.

Example 2.3.1. Find solutions to the equation

$$y'' + 5y' + 6y = 0. \tag{2.3.1}$$

Solution: We try to find solutions to this equation using simple test functions. For example, it is clear that power functions $y = t^n$ won't work, since the equation

$$n(n-1)\, t^{(n-2)} + 5n\, t^{(n-1)} + 6\, t^n = 0$$

cannot be satisfied for all $t \in \mathbb{R}$. We obtained, instead, a condition on t. This rules out power functions. A key insight is to try with a test function having a derivative proportional to the original function,

$$y'(t) = r\, y(t).$$

Such function would be simplified from the equation. For example, we try now with the test function $y(t) = e^{rt}$. If we introduce this function in the differential equation we get

$$(r^2 + 5r + 6)\, e^{rt} = 0 \quad \Leftrightarrow \quad r^2 + 5r + 6 = 0. \tag{2.3.2}$$

We have eliminated the exponential and any t dependence from the differential equation, and now the equation is a condition on the constant r. So we look for the appropriate values of r, which are the roots of a polynomial degree two,

$$r_\pm = \frac{1}{2}\big(-5 \pm \sqrt{25-24}\big) = \frac{1}{2}(-5 \pm 1) \quad \Rightarrow \quad \begin{cases} r_+ = -2, \\ r_- = -3. \end{cases}$$

We have obtained two different roots, which implies we have two different solutions,

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

These solutions are not proportional to each other, so the are fundamental solutions to the differential equation in (2.3.1). Therefore, Theorem 2.1.7 in § 2.1 implies that we have found all possible solutions to the differential equation, and they are given by

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}. \tag{2.3.3}$$

◁

From the example above we see that this idea will produce fundamental solutions to all constant coefficients homogeneous equations having associated polynomials with two different roots. Such polynomial play an important role to find solutions to differential equations as the one above, so we give such polynomial a name.

Definition 2.3.1. *The* ***characteristic polynomial*** *and* ***characteristic equation*** *of the second order linear homogeneous equation with constant coefficients*

$$y'' + a_1 y' + a_0 y = 0,$$

are given by

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

As we saw in Example 2.3.1, the roots of the characteristic polynomial are crucial to express the solutions of the differential equation above. The characteristic polynomial is a second degree polynomial with real coefficients, and the general expression for its roots is

$$r_\pm = \frac{1}{2}\Big(-a_1 \pm \sqrt{a_1^2 - 4a_0}\Big).$$

If the discriminant $(a_1^2 - 4a_0)$ is positive, zero, or negative, then the roots of p are different real numbers, only one real number, or a complex-conjugate pair of complex numbers. For each case the solution of the differential equation can be expressed in different forms.

Theorem 2.3.2 (Constant Coefficients)**.** *If* $r_\pm$ *are the roots of the characteristic polynomial to the second order linear homogeneous equation with constant coefficients*

$$y'' + a_1 y' + a_0 y = 0, \tag{2.3.4}$$

and if c_+, c_- *are arbitrary constants, then the following statements hold true.*

(a) If $r_+ \neq r_-$*, real or complex, then the general solution of Eq. (2.3.4) is given by*

$$y_{\rm gen}(t) = c_+ e^{r_+ t} + c_- e^{r_- t}.$$

(b) If $r_+ = r_- = r_0 \in \mathbb{R}$*, then the general solution of Eq. (2.3.4) is given by*

$$y_{\rm gen}(t) = c_+ e^{r_0 t} + c_- te^{r_0 t}.$$

Furthermore, given real constants t_0*,* y_0 *and* y_1*, there is a unique solution to the initial value problem given by Eq. (2.3.4) and the initial conditions* $y(t_0) = y_0$ *and* $y'(t_0) = y_1$*.*

Remarks:

(a) The proof is to guess that functions $y(t) = e^{rt}$ must be solutions for appropriate values of the exponent constant r, the latter being roots of the characteristic polynomial. When the characteristic polynomial has two different roots, Theorem 2.1.7 says we have all solutions. When the root is repeated we use the reduction of order method to find a second solution not proportional to the first one.

(b) At the end of the section we show a proof where we construct the fundamental solutions y_1, y_2 without guessing them. We do not need to use Theorem 2.1.7 in this second proof, which is based completely in a generalization of the reduction of order method.

Proof of Theorem 2.3.2: We guess that particular solutions to Eq. 2.3.4 must be exponential functions of the form $y(t) = e^{rt}$, because the exponential will cancel out from the equation and only a condition for r will remain. This is what happens,

$$r^2 e^{rt} + a_1 e^{rt} + a_0 e^{rt} = 0 \quad \Rightarrow \quad r^2 + a_1 r + a_0 = 0.$$

The second equation says that the appropriate values of the exponent are the root of the characteristic polynomial. We now have two cases. If $r_+ \neq r_-$ then the solutions

$$y_+(t) = e^{r_+ t}, \quad y_-(t) = e^{r_- t},$$

are linearly independent, so the general solution to the differential equation is

$$y_{\rm gen}(t) = c_+ e^{r_+ t} + c_- e^{r_- t}.$$

If $r_+ = r_- = r_0$, then we have found only one solution $y_+(t) = e^{r_0 t}$, and we need to find a second solution not proportional to y_+. This is what the reduction of order method is perfect for. We write the second solution as

$$y_-(t) = v(t)\, y_+(t) \quad \Rightarrow \quad y_-(t) = v(t)\, e^{r_0 t},$$

and we put this expression in the differential equation (2.3.4),

$$\big(v'' + 2r_0 v' + v r_0^2\big)\, e^{r_0 t} + \big(v' + r_0 v\big)\, a_1 e^{r_0 t} + a_0 v\, e^{r_0 t} = 0.$$

We cancel the exponential out of the equation and we reorder terms,

$$v'' + (2r_0 + a_1)\, v' + (r_0^2 + a_1 r_0 + a_0)\, v = 0.$$

We now need to use that r_0 is a root of the characteristic polynomial, $r_0^2 + a_1 r_0 + a_0 = 0$, so the last term in the equation above vanishes. But we also need to use that the root r_0 is repeated,

$$r_0 = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0} = -\frac{a_1}{2} \quad \Rightarrow \quad 2r_0 + a_1 = 0.$$

The equation on the right side above implies that the second term in the differential equation for v vanishes. So we get that

$$v'' = 0 \quad \Rightarrow \quad v(t) = c_1 + c_2 t$$

and the second solution is $y_-(t) = (c_1 + c_2 t)\, y_+(t)$. If we choose the constant $c_2 = 0$, the function y_- is proportional to y_+. So we definitely want $c_2 \neq 0$. The other constant, c_1, only adds a term proportional to y_+, we can choose it zero. So the simplest choice is $c_1 = 0$, $c_2 = 1$, and we get the fundamental solutions

$$y_+(t) = e^{r_0 t}, \qquad y_-(t) = t\, e^{r_0 t}.$$

So the general solution for the repeated root case is

$$y_{\text{gen}}(t) = c_+\, e^{r_0 t} + c_- t\, e^{r_0 t}.$$

The furthermore part follows from solving a 2×2 linear system for the unknowns c_+ and c_-. The initial conditions for the case $r_+ \neq r_-$ are the following,

$$y_0 = c_+\, e^{r_+ t_0} + c_-\, e^{r_- t_0}, \qquad y_1 = r_+ c_+\, e^{r_+ t_0} + r_- c_-\, e^{r_- t_0}.$$

It is not difficult to verify that this system is always solvable and the solutions are

$$c_+ = -\frac{(r_- y_0 - y_1)}{(r_+ - r_-)\, e^{r_+ t_0}}, \qquad c_- = \frac{(r_+ y_0 - y_1)}{(r_+ - r_-)\, e^{r_- t_0}}.$$

The initial conditions for the case $r_- = r_- = r_0$ are the following,

$$y_0 = (c_+ + c_- t_0)\, e^{r_0 t_0}, \qquad y_1 = c_-\, e^{r_0 t_0} + r_0 (c_+ + c_- t_0)\, e^{r_0 t_0}.$$

It is also not difficult to verify that this system is always solvable and the solutions are

$$c_+ = \frac{y_0 + t_0 (r_0 y_0 - y_1)}{e^{r_0 t_0}}, \qquad c_- = -\frac{(r_0 y_0 - y_0)}{e^{r_0 t_0}}.$$

This establishes the Theorem. □

Example 2.3.2. Find the solution y of the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Solution: We know that the general solution of the differential equation above is

$$y_{\text{gen}}(t) = c_+ e^{-2t} + c_- e^{-3t}.$$

We now find the constants c_+ and c_- that satisfy the initial conditions above,

$$\left.\begin{aligned} 1 &= y(0) = c_+ + c_- \\ -1 &= y'(0) = -2c_+ - 3c_- \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} c_+ = 2, \\ c_- = -1. \end{cases}$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

◁

Example 2.3.3. Find the general solution y_{gen} of the differential equation

$$2y'' - 3y' + y = 0.$$

Solution: We look for every solutions of the form $y(t) = e^{rt}$, where r is solution of the characteristic equation

$$2r^2 - 3r + 1 = 0 \quad \Rightarrow \quad r = \frac{1}{4}\big(3 \pm \sqrt{9-8}\big) \quad \Rightarrow \quad \begin{cases} r_+ = 1, \\ r_- = \dfrac{1}{2}. \end{cases}$$

Therefore, the general solution of the equation above is

$$y_{\text{gen}}(t) = c_+ e^t + c_- e^{t/2}.$$

◁

Example 2.3.4. Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{5}{3}.$$

Solution: The characteristic polynomial is $p(r) = 9r^2 + 6r + 1$, with roots given by

$$r_\pm = \frac{1}{18}\big(-6 \pm \sqrt{36-36}\big) \quad \Rightarrow \quad r_+ = r_- = -\frac{1}{3}.$$

Theorem 2.3.2 says that the general solution has the form

$$y_{\text{gen}}(t) = c_+ \, e^{-t/3} + c_- t \, e^{-t/3}.$$

We need to compute the derivative of the expression above to impose the initial conditions,

$$y'_{\text{gen}}(t) = -\frac{c_+}{3}\, e^{-t/3} + c_- \Big(1 - \frac{t}{3}\Big)\, e^{-t/3},$$

then, the initial conditions imply that

$$\left.\begin{aligned} 1 &= y(0) = c_+, \\ \frac{5}{3} &= y'(0) = -\frac{c_+}{3} + c_- \end{aligned}\right\} \quad \Rightarrow \quad c_+ = 1, \qquad c_- = 2.$$

So, the solution to the initial value problem above is: $y(t) = (1 + 2t)\, e^{-t/3}$. ◁

Example 2.3.5. Find the general solution y_{gen} of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: We first find the roots of the characteristic polynomial,

$$r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(2 \pm \sqrt{4-24}\big) \quad \Rightarrow \quad r_\pm = 1 \pm i\sqrt{5}.$$

Since the roots of the characteristic polnomial are different, Theorem 2.3.2 says that the general solution of the differential equation above, which includes complex-valued solutions, can be written as follows,

$$y_{\rm gen}(t) = \tilde{c}_{+}\, e^{(1+i\sqrt{5})t} + \tilde{c}_{-}\, e^{(1-i\sqrt{5})t}, \qquad \tilde{c}_{+}, \tilde{c}_{-} \in \mathbb{C}.$$

$\lhd$

2.3.2. Real Solutions for Complex Roots. We study in more detail the solutions to the differential equation (2.3.4) in the case that the characteristic polynomial has complex roots. Since these roots have the form

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0},$$

the roots are complex-valued in the case $a_1^2 - 4a_0 < 0$. We use the notation

$$r_{\pm} = \alpha \pm i\beta, \quad \text{with} \quad \alpha = -\frac{a_1}{2}, \quad \beta = \sqrt{a_0 - \frac{a_1^2}{4}}.$$

The fundamental solutions in Theorem 2.3.2 are the complex-valued functions

$$\tilde{y}_{+} = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_{-} = e^{(\alpha-i\beta)t}.$$

The general solution constructed from these solutions is

$$y_{\rm gen}(t) = \tilde{c}_{+}\, e^{(\alpha+i\beta)t} + \tilde{c}_{-}\, e^{(\alpha-i\beta)t}, \qquad \tilde{c}_{+},\ \tilde{c}_{-} \in \mathbb{C}.$$

This formula for the general solution includes real valued and complex valued solutions. But it is not so simple to single out the real valued solutions. Knowing the real valued solutions could be important in physical applications. If a physical system is described by a differential equation with real coefficients, more often than not one is interested in finding real valued solutions. For that reason we now provide a new set of fundamental solutions that are real valued. Using real valued fundamental solution is simple to separate all real valued solutions from the complex valued ones.

Theorem 2.3.3 (Real Valued Fundamental Solutions)**.** *If the differential equation*

$$y'' + a_1\, y' + a_0\, y = 0, \tag{2.3.5}$$

where a_1, a_0 are real constants, has characteristic polynomial with complex roots $r_{\pm} = \alpha \pm i\beta$ and complex valued fundamental solutions

$$\tilde{y}_{+}(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_{-}(t) = e^{(\alpha-i\beta)t},$$

then the equation also has real valued fundamental solutions given by

$$y_{+}(t) = e^{\alpha t}\cos(\beta t), \qquad y_{-}(t) = e^{\alpha t}\sin(\beta t).$$

Proof of Theorem 2.3.3: We start with the complex valued fundamental solutions

$$\tilde{y}_{+}(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_{-}(t) = e^{(\alpha-i\beta)t}.$$

We take the function $\tilde{y}_{+}$ and we use a property of complex exponentials,

$$\tilde{y}_{+}(t) = e^{(\alpha+i\beta)t} = e^{\alpha t}\, e^{i\beta t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big),$$

where on the last step we used Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Repeat this calculation for y_{-} we get,

$$\tilde{y}_{+}(t) = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big), \quad \tilde{y}_{-}(t) = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big).$$

If we recall the superposition property of linear homogeneous equations, Theorem 2.1.5, we know that any linear combination of the two solutions above is also a solution of the differential equation (2.3.6), in particular the combinations

$$y_+(t) = \frac{1}{2}\big(\tilde{y}_+(t) + \tilde{y}_-(t)\big), \quad y_-(t) = \frac{1}{2i}\big(\tilde{y}_+(t) - \tilde{y}_-(t)\big).$$

A straightforward computation gives

$$y_+(t) = e^{\alpha t}\cos(\beta t), \qquad y_-(t) = e^{\alpha t}\sin(\beta t).$$

This establishes the Theorem. □

Example 2.3.6. Find the real valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: We already found the roots of the characteristic polynomial, but we do it again,

$$r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(2 \pm \sqrt{4-24}\big) \quad \Rightarrow \quad r_\pm = 1 \pm i\sqrt{5}.$$

So the complex valued fundamental solutions are

$$\tilde{y}_+(t) = e^{(1+i\sqrt{5})\,t}, \quad \tilde{y}_-(t) = e^{(1-i\sqrt{5})\,t}.$$

Theorem **??** says that real valued fundamental solutions are given by

$$y_+(t) = e^t\cos(\sqrt{5}t), \quad y_-(t) = e^t\sin(\sqrt{5}t).$$

So the real valued general solution is given by

$$y_{\text{gen}}(t) = \big(c_+\cos(\sqrt{5}\,t) + c_-\sin(\sqrt{5}\,t)\big)\,e^t, \quad c_+,\ c_- \in \mathbb{R}.$$

◁

Remark: Sometimes it is difficult to remember the formula for real valued solutions. One way to obtain those solutions without remembering the formula is to start repeat the proof of Theorem 2.3.3. Start with the complex valued solution $\tilde{y}_+$ and use the properties of the complex exponential,

$$\tilde{y}_+(t) = e^{(1+i\sqrt{5})t} = e^t\,e^{i\sqrt{5}t} = e^t\,\big(\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)\big).$$

The real valued fundamental solutions are the real and imaginary parts in that expression.

Example 2.3.7. Find real valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution: The roots of the characteristic polynomial $p(r) = r^2 + 2r + 6$ are

$$r_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4-24}\big] = \frac{1}{2}\big[-2 \pm \sqrt{-20}\big] \quad \Rightarrow \quad r_\pm = -1 \pm i\sqrt{5}.$$

These are complex-valued roots, with

$$\alpha = -1, \qquad \beta = \sqrt{5}.$$

Real-valued fundamental solutions are

$$y_1(t) = e^{-t}\,\cos(\sqrt{5}\,t), \qquad y_2(t) = e^{-t}\,\sin(\sqrt{5}\,t).$$

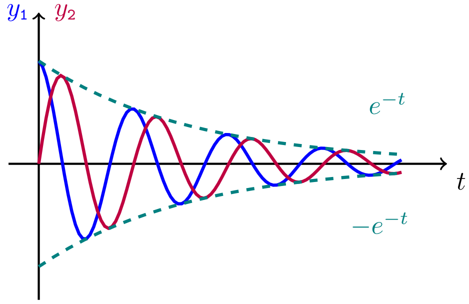


FIGURE 1. Solutions from Ex. 2.3.7.

Second order differential equations with characteristic polynomials having complex roots, like the one in this example, describe physical processes related to damped oscillations. An example from physics is a pendulums with friction. ◁

Example 2.3.8. Find the real valued general solution of $y'' + 5\,y = 0$.

Solution: The characteristic polynomial is $p(r) = r^2 + 5$, with roots $r_\pm = \pm\sqrt{5}\,i$. In this case $\alpha = 0$, and $\beta = \sqrt{5}$. Real valued fundamental solutions are

$$y_+(t) = \cos(\sqrt{5}\,t), \qquad y_-(t) = \sin(\sqrt{5}\,t).$$

The real valued general solution is

$$y_{\rm gen}(t) = c_+\,\cos(\sqrt{5}\,t) + c_-\,\sin(\sqrt{5}\,t), \qquad c_+, c_- \in \mathbb{R}.$$

◁

Remark: Physical processes that oscillate in time without dissipation could be described by differential equations like the one in this example.

2.3.3. Constructive Proof of Theorem 2.3.2.

We now present an alternative proof for Theorem 2.3.2 that does not involve guessing the fundamental solutions of the equation. Instead, we construct these solutions using a generalization of the reduction of order method.

Proof of Theorem 2.3.2: The proof has two main parts: First, we transform the original equation into an equation simpler to solve for a new unknown; second, we solve this simpler problem.

In order to transform the problem into a simpler one, we express the solution y as a product of two functions, that is, $y(t) = u(t)v(t)$. Choosing v in an appropriate way the equation for u will be simpler to solve than the equation for y. Hence,

$$y = uv \quad \Rightarrow \quad y' = u'v + v'u \quad \Rightarrow \quad y'' = u''v + 2u'v' + v''u.$$

Therefore, Eq. (2.3.4) implies that

$$(u''v + 2u'v' + v''u) + a_1\,(u'v + v'u) + a_0\,uv = 0,$$

that is,

$$\Big[u'' + \Big(a_1 + 2\frac{v'}{v}\Big)\,u' + a_0\,u\Big]v + (v'' + a_1\,v')\,u = 0. \tag{2.3.6}$$

We now choose the function v such that

$$a_1 + 2\frac{v'}{v} = 0 \quad \Leftrightarrow \quad \frac{v'}{v} = -\frac{a_1}{2}. \tag{2.3.7}$$

We choose a simple solution of this equation, given by

$$v(t) = e^{-a_1 t/2}.$$

Having this expression for v one can compute v' and v'', and it is simple to check that

$$v'' + a_1 v' = -\frac{a_1^2}{4}\, v. \tag{2.3.8}$$

Introducing the first equation in (2.3.7) and Eq. (2.3.8) into Eq. (2.3.6), and recalling that v is non-zero, we obtain the simplified equation for the function u, given by

$$u'' - k\, u = 0, \qquad k = \frac{a_1^2}{4} - a_0. \tag{2.3.9}$$

Eq. (2.3.9) for u is simpler than the original equation (2.3.4) for y since in the former there is no term with the first derivative of the unknown function.

In order to solve Eq. (2.3.9) we repeat the idea followed to obtain this equation, that is, express function u as a product of two functions, and solve a simple problem of one of the functions. We first consider the harder case, which is when $k \neq 0$. In this case, let us express $u(t) = e^{\sqrt{k}t}\, w(t)$. Hence,

$$u' = \sqrt{k} e^{\sqrt{k}t}\, w + e^{\sqrt{k}t}\, w' \quad \Rightarrow \quad u'' = k e^{\sqrt{k}t}\, w + 2\sqrt{k} e^{\sqrt{k}t}\, w' + e^{\sqrt{k}t}\, w''.$$

Therefore, Eq. (2.3.9) for function u implies the following equation for function w

$$0 = u'' - ku = e^{\sqrt{k}t}\, (2\sqrt{k}\, w' + w'') \quad \Rightarrow \quad w'' + 2\sqrt{k} w' = 0.$$

Only derivatives of w appear in the latter equation, so denoting $x(t) = w'(t)$ we have to solve a simple equation

$$x' = -2\sqrt{k}\, x \quad \Rightarrow \quad x(t) = x_0 e^{-2\sqrt{k}t}, \qquad x_0 \in \mathbb{R}.$$

Integrating we obtain w as follows,

$$w' = x_0 e^{-2\sqrt{k}t} \quad \Rightarrow \quad w(t) = -\frac{x_0}{2\sqrt{k}} e^{-2\sqrt{k}t} + c_0.$$

renaming $c_1 = -x_0/(2\sqrt{k})$, we obtain

$$w(t) = c_1 e^{-2\sqrt{k}t} + c_0 \quad \Rightarrow \quad u(t) = c_0 e^{\sqrt{k}t} + c_1 e^{-\sqrt{k}t}.$$

We then obtain the expression for the solution $y = uv$, given by

$$y(t) = c_0 e^{(-\frac{a_1}{2} + \sqrt{k})t} + c_1 e^{(-\frac{a_1}{2} - \sqrt{k})t}.$$

Since $k = (a_1^2/4 - a_0)$, the numbers

$$r_{\pm} = -\frac{a_1}{2} \pm \sqrt{k} \quad \Leftrightarrow \quad r_{\pm} = \frac{1}{2}\Big(-a_1 \pm \sqrt{a_1^2 - 4a_0}\Big)$$

are the roots of the characteristic polynomial

$$r^2 + a_1\, r + a_0 = 0,$$

we can express all solutions of the Eq. (2.3.4) as follows

$$y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}, \qquad k \neq 0.$$

Finally, consider the case $k = 0$. Then, Eq. (2.3.9) is simply given by

$$u'' = 0 \quad \Rightarrow \quad u(t) = (c_0 + c_1 t) \qquad c_0, c_1 \in \mathbb{R}.$$

Then, the solution y to Eq. (2.3.4) in this case is given by

$$y(t) = (c_0 + c_1 t)\, e^{-a_1 t/2}.$$

Since $k = 0$, the characteristic equation $r^2 + a_1\, r + a_0 = 0$ has only one root $r_+ = r_- = -a_1/2$, so the solution y above can be expressed as

$$y(t) = (c_0 + c_1 t)\, e^{r_+ t}, \qquad k = 0.$$

The Furthermore part is the same as in Theorem 2.3.2. This establishes the Theorem. ☐

Notes.

(a) In the case that the characteristic polynomial of a differential equation has repeated roots there is an interesting argument to guess the solution y_-. The idea is to take a particular type of limit in solutions of differential equations with complex valued roots.

Consider the equation in (2.3.4) with a characteristic polynomial having complex valued roots given by $r_\pm = \alpha \pm i\beta$, with

$$\alpha = -\frac{a_1}{2}, \qquad \beta = \sqrt{a_0 - \frac{a_1^2}{4}}.$$

Real valued fundamental solutions in this case are given by

$$\hat{y}_+ = e^{\alpha\,t}\,\cos(\beta t), \quad \hat{y}_- = e^{\alpha\,t}\,\sin(\beta t).$$

We now study what happen to these solutions $\hat{y}_+$ and $\hat{y}_-$ in the following limit: The variable t is held constant, α is held constant, and $\beta \to 0$. The last two conditions are conditions on the equation coefficients, a_1, a_0. For example, we fix a_1 and we vary $a_0 \to a_1^2/4$ from above.

Since $\cos(\beta t) \to 1$ as $\beta \to 0$ with t fixed, then keeping α fixed too, we obtain

$$\hat{y}_+(t) = e^{\alpha\,t}\cos(\beta t) \;\longrightarrow\; e^{\alpha\,t} = y_+(t).$$

Since $\dfrac{\sin(\beta t)}{\beta t} \to 1$ as $\beta \to 0$ with t constant, that is, $\sin(\beta t) \to \beta t$, we conclude that

$$\frac{\hat{y}_-(t)}{\beta} = \frac{\sin(\beta t)}{\beta}\,e^{\alpha\,t} = \frac{\sin(\beta t)}{\beta t}\,t\,e^{\alpha\,t} \;\longrightarrow\; t\,e^{\alpha\,t} = y_-(t).$$

The calculation above says that the function $\hat{y}_-/\beta$ is close to the function $y_-(t) = t\,e^{\alpha\,t}$ in the limit $\beta \to 0$, t held constant. This calculation provides a candidate, $y_-(t) = t\,y_+(t)$, of a solution to Eq. (2.3.4). It is simple to verify that this candidate is in fact solution of Eq. (2.3.4). Since y_- is not proportional to y_+, one then concludes the functions y_+, y_- are a fundamental set for the differential equation in (2.3.4) in the case the characteristic polynomial has repeated roots.

(b) Brief Review of Complex Numbers.

- Complex numbers have the form $z = a + ib$, where $i^2 = -1$.
- The complex conjugate of z is the number $\overline{z} = a - ib$.
- $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$ are the real and imaginary parts of z
- Hence: $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$, and $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$.
- The exponential of a complex number is defined as

$$e^{a+ib} = \sum_{n=0}^{\infty} \frac{(a+ib)^n}{n!}.$$

  In particular holds $e^{a+ib} = e^a\,e^{ib}$.
- Euler's formula: $e^{ib} = \cos(b) + i\sin(b)$.
- Hence, a complex number of the form e^{a+ib} can be written as

$$e^{a+ib} = e^a\big(\cos(b) + i\sin(b)\big), \quad e^{a-ib} = e^a\big(\cos(b) - i\sin(b)\big).$$

- From e^{a+ib} and e^{a-ib} we get the real numbers

$$\frac{1}{2}\big(e^{a+ib} + e^{a-ib}\big) = e^a\,\cos(b), \quad \frac{1}{2i}\big(e^{a+ib} - e^{a-ib}\big) = e^a\,\sin(b).$$

2.3.4. Exercises.

2.3.1.- Consider the differential equation

$$y'' - 7y' + 12y = 0.$$

(a) Find the roots $r_\pm$ of the characteristic polynomial associated with the differential equation.

(b) Use the roots $r_\pm$ above to find fundamental solutions $y_\pm$ of the differential equation.

(c) Solve the differential equation above with initial conditions

$$y(0) = 1, \quad y'(0) = -1.$$

2.3.2.- Consider the differential equation

$$y'' - 8y' + 25y = 0.$$

(a) Find the roots $r_\pm$ of the characteristic polynomial associated with the differential equation.

(b) Use the roots $r_\pm$ above to find **real valued** fundamental solutions $y_\pm$ of the differential equation.

(c) Solve the differential equation above with initial conditions

$$y(0) = 2, \quad y'(0) = 2.$$

2.3.3.- Consider the differential equation

$$y'' - 6y' + 9y = 0.$$

(a) Find the roots $r_\pm$ of the characteristic polynomial associated with the differential equation.

(b) Use the roots $r_\pm$ above to find **real valued** fundamental solutions $y_\pm$ of the differential equation.

(c) Solve the differential equation above with initial conditions

$$y(0) = 1, \quad y'(0) = 2.$$

2.3.4.- ***** Consider the differential equation

$$y'' - 4y' + 4y = 0.$$

(a) Find one solution of the form $y_1(t) = e^{rt}$.

(b) Use the **reduction order method** to find a second solution

$$y_2(t) = v(t)\, y_1(t).$$

First find the differential equation satisfied by $v(t)$.

(c) Find all solutions $v(t)$ of the differential equation in part (b).

(d) Choose a function v such that the associated solution y_2 does not contain any term proportional to y_1.

2.4. Euler Equidimensional Equation

Second order linear equations with variable coefficients are in general more difficult to solve than equations with constant coefficients. But the Euler equidimensional equation is an exception to this rule. The same ideas we used to solve second order linear equations with constant coefficients can be used to solve Euler's equidimensional equation. Moreover, there is a transformation that converts Euler's equation into a linear equation.

2.4.1. The Roots of the Indicial Polynomial. The Euler equidimensional equation appears, for example, when one solves the two-dimensional Laplace equation in polar coordinates. This happens if one tries to find the electrostatic potential of a two-dimensional charge configuration having circular symmetry. The Euler equation is simple to recongnize—the coefficient of each term in the equation is a power of the independent variable that matches the order of the derivative in that term.

Definition 2.4.1. *The* ***Euler equidimensional equation*** *for the unknown y with singular point at $t_0 \in \mathbb{R}$ is given by the equation below, where a_1 and a_0 are constants,*

$$(t-t_0)^2\, y'' + a_1\,(t-t_0)\, y' + a_0\, y = 0.$$

Remarks:

(a) This equation is also called Cauchy equidimensional equation, Cauchy equation, Cauchy-Euler equation, or simply Euler equation. As George Simmons says in [10], "Euler studies were so extensive that many mathematicians tried to avoid confusion by naming subjects after the person who first studied them after Euler."

(b) The equation is called equidimensional because if the variable t has any physical dimensions, then the terms with $(t-t_0)^n \dfrac{d^n}{dt^n}$, for any nonnegative integer n, are actually dimensionless.

(c) The exponential functions $y(t) = e^{rt}$ are not solutions of the Euler equation. Just introduce such a function into the equation, and it is simple to show that there is no constant r such that the exponential is solution.

(d) The particular case $t_0 = 0$ is

$$t^2\, y'' + p_0\, t\, y' + q_0\, y = 0.$$

We now summarize what is known about solutions of the Euler equation.

Theorem 2.4.2 (Euler Equation)**.** *Consider the Euler equidimensional equation*

$$(t-t_0)^2\, y'' + a_1\,(t-t_0)\, y' + a_0\, y = 0, \qquad t > t_0, \tag{2.4.1}$$

where a_1, a_0, and t_0 are real constants, and denote by $r_\pm$ the roots of the indicial polynomial $p(r) = r(r-1) + a_1 r + a_0$.

(a) If $r_+ \neq r_-$, real or complex, then the general solution of Eq. (2.4.1) is given by

$$y_{gen}(t) = c_+ (t-t_0)^{r_+} + c_- (t-t_0)^{r_-}, \quad t > t_0, \quad c_+,\ c_- \in \mathbb{R}.$$

(b) If $r_+ = r_- = r_0 \in \mathbb{R}$, then the general solution of Eq. (2.4.1) is given by

$$y_{gen}(t) = c_+\,(t-t_0)^{r_0} + c_-\,(t-t_0)^{r_0}\,\ln(t-t_0), \quad t > t_0, \quad c_+,\ c_- \in \mathbb{R}.$$

Furthermore, given real constants $t_1 > t_0$, y_0 *and* y_1, *there is a unique solution to the initial value problem given by Eq. (2.4.1) and the initial conditions*

$$y(t_1) = y_0, \qquad y'(t_1) = y_1.$$

Remark: We have restricted to a domain with $t > t_0$. Similar results hold for $t < t_0$. In fact one can prove the following: If a solution y has the value $y(t - t_0)$ at $t - t_0 > 0$, then the function $\tilde{y}$ defined as $\tilde{y}(t - t_0) = y(-(t - t_0))$, for $t - t_0 < 0$ is solution of Eq. (2.4.1) for $t - t_0 < 0$. For this reason the solution for $t \neq t_0$ is sometimes written in the literature, see [3] § 5.4, as follows,

$$y_{\text{gen}}(t) = c_+ |t - t_0|^{r_+} + c_- |t - t_0|^{r_-}, \quad r_+ \neq r_-,$$
$$y_{\text{gen}}(t) = c_+ \, |t - t_0|^{r_0} + c_- \, |t - t_0|^{r_0} \, \ln |t - t_0|, \quad r_+ = r_- = r_0.$$

However, when solving an initial value problem, we need to pick the domain that contains the initial data point t_1. This domain will be a subinterval in either $(-\infty, t_0)$ or (t_0, ∞). For simplicity, in these notes we choose the domain (t_0, ∞).

The proof of this theorem closely follows the ideas to find all solutions of second order linear equations with constant coefficients, Theorem 2.3.2, in § 2.3. In that case we found fundamental solutions to the differential equation

$$y'' + a_1 \, y' + a_0 \, y = 0,$$

and then we recalled Theorem 2.1.7, which says that any other solution is a linear combination of a fundamental solution pair. In the case of constant coefficient equations, we looked for fundamental solutions of the form $y(t) = e^{rt}$, where the constant r was a root of the characteristic polynomial

$$r^2 + a_1 r + a_0 = 0.$$

When this polynomial had two different roots, $r_+ \neq r_-$, we got the fundamental solutions

$$y_+(t) = e^{r_+ t}, \qquad y_-(t) = e^{r_- t}.$$

When the root was repeated, $r_+ = r_- = r_0$, we used the reduction order method to get the fundamental solutions

$$y_+(t) = e^{r_0 t}, \qquad y_-(t) = t \, e^{r_0 t}.$$

Well, the proof of Theorem 2.4.2 closely follows this proof, *replacing the exponential function by power functions.*

Proof of Theorem 2.4.2: For simplicity we consider the case $t_0 = 0$. The general case $t_0 \neq 0$ follows from the case $t_0 = 0$ replacing t by $(t - t_0)$. So, consider the equation

$$t^2 \, y'' + a_1 \, t \, y' + a_0 \, y = 0, \qquad t > 0.$$

We look for solutions of the form $y(t) = t^r$, because power functions have the property that

$$y' = r \, t^{r-1} \quad \Rightarrow \quad t \, y' = r \, t^r.$$

A similar property holds for the second derivative,

$$y'' = r(r-1) \, t^{r-2} \quad \Rightarrow \quad t^2 \, y'' = r(r-1) \, t^r.$$

When we introduce this function into the Euler equation we get an algebraic equation for r,

$$\big[r(r-1) + a_1 r + a_0 \big] \, t^r = 0 \quad \Leftrightarrow \quad r(r-1) + a_1 r + a_0 = 0.$$

The constant r must be a root of the indicial polynomial

$$p(r) = r(r-1) + a_1 r + a_0.$$

This polynomial is sometimes called the Euler characteristic polynomial. So we have two possibilities. If the roots are different, $r_+ \neq r_-$, we get the fundamental solutions

$$y_+(t) = t^{r_+}, \qquad y_-(t) = t^{r_-}.$$

If we have a repeated root $r_+ = r_- = r_0$, then one solution is $y_+(t) = t^{r_0}$. To obtain the second solution we use the reduction order method. Since we have one solution to the equation, y_+, the second solution is

$$y_-(t) = v(t)\, y_+(t) \quad \Rightarrow \quad y_-(t) = v(t)\, t^{r_0}.$$

We need to compute the first two derivatives of y_-,

$$y_-' = r_0 v\, t^{r_0-1} + v'\, t^{r_0}, \qquad y_-'' = r_0(r_0-1)v\, t^{r_0-2} + 2r_0 v'\, t^{r_0-1} + v''\, t^{r_0}.$$

We now put these expressions for y_-, y_-' and y_-'' into the Euler equation,

$$t^2\left(r_0(r_0-1)v\, t^{r_0-2} + 2r_0 v'\, t^{r_0-1} + v''\, t^{r_0}\right) + a_1 t\left(r_0 v\, t^{r_0-1} + v'\, t^{r_0}\right) + a_0\, v\, t^{r_0} = 0.$$

Let us reorder terms in the following way,

$$v''\, t^{r_0+2} + (2r_0 + a_1)\, v'\, t^{r_0+1} + \big[r_0(r_0-1) + a_1 r_0 + a_0\big]\, v\, t^{r_0} = 0.$$

We now need to recall that r_0 is both a root of the indicial polynomial,

$$r_0(r_0-1) + a_1 r_0 + a_0 = 0$$

and r_0 is a repeated root, that is $(a_1-1)^2 = 4a_0$, hence

$$r_0 = -\frac{(a_1-1)}{2} \quad \Rightarrow \quad 2r_0 + a_1 = 1.$$

Using these two properties of r_0 in the Euler equation above, we get the equation for v,

$$v''\, t^{r_0+2} + v'\, t^{r_0+1} = 0 \quad \Rightarrow \quad v''\, t + v' = 0.$$

This is a first order equation for $w = v'$,

$$w'\, t + w = 0 \quad \Rightarrow \quad (t\, w)' = 0 \quad \Rightarrow \quad w(t) = \frac{w_0}{t}.$$

We now integrate one last time to get function v,

$$v' = \frac{w_0}{t} \quad \Rightarrow \quad v(t) = w_0 \ln(t) + v_0.$$

So the second solution to the Euler equation in the case of repeated roots is

$$y_-(t) = \big(w_0 \ln(t) + v_0\big)\, t^{r_0} \quad \Rightarrow \quad y_-(t) = w_0 t^{r_0} \ln(t) + v_0\, y_+(t).$$

It is clear we can choose $v_0 = 0$ and $w_0 = 1$ to get

$$y_-(t) = t^{r_0} \ln(t).$$

We got fundamental solutions for all roots of the indicial polynomial, and their general solutions follow from Theorem 2.1.7 in § 2.1. This establishes the Theorem. □

Example 2.4.1. Find the general solution of the Euler equation below for $t > 0$,

$$t^2\, y'' + 4t\, y' + 2\, y = 0.$$

Solution: We look for solutions of the form $y(t) = t^r$, which implies that

$$t\, y'(t) = r\, t^r, \qquad t^2\, y''(t) = r(r-1)\, t^r,$$

therefore, introducing this function y into the differential equation we obtain

$$\big[r(r-1) + 4r + 2\big]\, t^r = 0 \quad \Leftrightarrow \quad r(r-1) + 4r + 2 = 0.$$

The solutions are computed in the usual way,

$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[-3 \pm \sqrt{9-8}\big] \quad \Rightarrow \quad \begin{cases} r_+ = -1 \\ r_- = -2. \end{cases}$$

So the general solution of the differential equation above is given by

$$y_{\text{gen}}(t) = c_+ \, t^{-1} + c_- \, t^{-2}.$$

◁

Remark: Both fundamental solutions in the example above diverge at $t = 0$.

Example 2.4.2. Find the general solution of the Euler equation below for $t > 0$,

$$t^2 \, y'' - 3t \, y' + 4 \, y = 0.$$

Solution: We look for solutions of the form $y(t) = t^r$, then the constant r must be solution of the Euler characteristic polynomial

$$r(r-1) - 3r + 4 = 0 \quad \Leftrightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad r_+ = r_- = 2.$$

Therefore, the general solution of the Euler equation for $t > 0$ in this case is given by

$$y_{\text{gen}}(t) = c_+ t^2 + c_- t^2 \ln(t).$$

◁

Example 2.4.3. Find the general solution of the Euler equation below for $t > 0$,

$$t^2 \, y'' - 3t \, y' + 13 \, y = 0.$$

Solution: We look for solutions of the form $y(t) = t^r$, which implies that

$$t \, y'(t) = r \, t^r, \qquad t^2 \, y''(t) = r(r-1) \, t^r,$$

therefore, introducing this function y into the differential equation we obtain

$$\big[r(r-1) - 3r + 13\big] \, t^r = 0 \quad \Leftrightarrow \quad r(r-1) - 3r + 13 = 0.$$

The solutions are computed in the usual way,

$$r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{-36}\big] \quad \Rightarrow \quad \begin{cases} r_+ = 2 + 3i \\ r_- = 2 - 3i. \end{cases}$$

So the general solution of the differential equation above is given by

$$y_{\text{gen}}(t) = c_+ \, t^{(2+3i)} + c_- \, t^{(2-3i)}. \tag{2.4.2}$$

◁

2.4.2. Real Solutions for Complex Roots.

We study in more detail the solutions to the Euler equation in the case that the indicial polynomial has complex roots. Since these roots have the form

$$r_\pm = -\frac{(a_1 - 1)}{2} \pm \frac{1}{2} \sqrt{(a_1 - 1)^2 - 4a_0},$$

the roots are complex-valued in the case $(p_0 - 1)^2 - 4q_0 < 0$. We use the notation

$$r_\pm = \alpha \pm i\beta, \quad \text{with} \quad \alpha = -\frac{(a_1 - 1)}{2}, \quad \beta = \sqrt{a_0 - \frac{(a_1 - 1)^2}{4}}.$$

The fundamental solutions in Theorem 2.4.2 are the complex-valued functions

$$\tilde{y}_+(t) = t^{(\alpha + i\beta)}, \qquad \tilde{y}_-(t) = t^{(\alpha - i\beta)}.$$

The general solution constructed from these solutions is

$$y_{\mathrm{gen}}(t) = \tilde{c}_{+}\, t^{(\alpha+i\beta)} + \tilde{c}_{-}\, t^{(\alpha-i\beta)}, \qquad \tilde{c}_{+},\ \tilde{c}_{-} \in \mathbb{C}.$$

This formula for the general solution includes real valued and complex valued solutions. But it is not so simple to single out the real valued solutions. Knowing the real valued solutions could be important in physical applications. If a physical system is described by a differential equation with real coefficients, more often than not one is interested in finding real valued solutions. For that reason we now provide a new set of fundamental solutions that are real valued. Using real valued fundamental solution is simple to separate all real valued solutions from the complex valued ones.

Theorem 2.4.3 (Real Valued Fundamental Solutions)**.** *If the differential equation*

$$(t-t_0)^2\, y'' + a_1 (t-t_0)\, y' + a_0\, y = 0, \qquad t > t_0, \tag{2.4.3}$$

where a_1, a_0, t_0 are real constants, has indicial polynomial with complex roots $r_{\pm} = \alpha \pm i\beta$ and complex valued fundamental solutions for $t > t_0$,

$$\tilde{y}_{+}(t) = (t-t_0)^{(\alpha+i\beta)}, \qquad \tilde{y}_{-}(t) = (t-t_0)^{(\alpha-i\beta)},$$

then the equation also has real valued fundamental solutions for $t > t_0$ given by

$$y_{+}(t) = (t-t_0)^{\alpha} \cos\big(\beta \ln(t-t_0)\big), \qquad y_{-}(t) = (t-t_0)^{\alpha} \sin\big(\beta \ln(t-t_0)\big).$$

Proof of Theorem 2.4.3: For simplicity consider the case $t_0 = 0$. Take the solutions

$$\tilde{y}_{+}(t) = t^{(\alpha+i\beta)}, \qquad \tilde{y}_{-}(t) = t^{(\alpha-i\beta)}.$$

Rewrite the power function as follows,

$$\tilde{y}_{+}(t) = t^{(\alpha+i\beta)} = t^{\alpha}\, t^{i\beta} = t^{\alpha}\, e^{\ln(t^{i\beta})} = t^{\alpha}\, e^{i\beta \ln(t)} \quad \Rightarrow \quad \tilde{y}_{+}(t) = t^{\alpha}\, e^{i\beta\ln(t)}.$$

A similar calculation yields

$$\tilde{y}_{-}(t) = t^{\alpha}\, e^{-i\beta\ln(t)}.$$

Recall now Euler formula for complex exponentials, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, then we get

$$\tilde{y}_{+}(t) = t^{\alpha}\, \big[\cos\big(\beta\ln(t)\big) + i\sin\big(\beta\ln(t)\big)\big], \qquad \tilde{y}_{-}(t) = t^{\alpha}\, \big[\cos\big(\beta\ln(t)\big) - i\sin\big(\beta\ln(t)\big)\big].$$

Since $\tilde{y}_{+}$ and $\tilde{y}_{-}$ are solutions to Eq. (2.4.3), so are the functions

$$y_1(t) = \frac{1}{2}\big[\tilde{y}_1(t) + \tilde{y}_2(t)\big], \qquad y_2(t) = \frac{1}{2i}\big[\tilde{y}_1(t) - \tilde{y}_2(t)\big].$$

It is not difficult to see that these functions are

$$y_{+}(t) = t^{\alpha}\cos\big(\beta\ln(t)\big), \qquad y_{-}(t) = t^{\alpha}\sin\big(\beta\ln(t)\big).$$

To prove the case having $t_0 \neq 0$, just replace t by $(t-t_0)$ on all steps above. This establishes the Theorem. □

Example 2.4.4. Find a real-valued general solution of the Euler equation below for $t > 0$,

$$t^2\, y'' - 3t\, y' + 13\, y = 0.$$

Solution: The indicial equation is $r(r-1) - 3r + 13 = 0$, with solutions

$$r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_{+} = 2 + 3i, \quad r_{-} = 2 - 3i.$$

A complex-valued general solution for $t > 0$ is,

$$y_{\mathrm{gen}}(t) = \tilde{c}_{+}\, t^{(2+3i)} + \tilde{c}_{-}\, t^{(2-3i)} \quad \tilde{c}_{+},\ \tilde{c}_{-} \in \mathbb{C}.$$

A real-valued general solution for $t > 0$ is

$$y_{\rm gen}(t) = c_{+}\, t^2 \cos\big(3\ln(t)\big) + c_{-}\, t^2 \sin\big(3\ln(t)\big), \qquad c_{+},\ c_{-} \in \mathbb{R}.$$

$\lhd$

2.4.3. Transformation to Constant Coefficients. Theorem 2.4.2 shows that $y(t) = t^{r_\pm}$, where $r_\pm$ are roots of the indicial polynomial, are solutions to the Euler equation

$$t^2\, y'' + a_1 t\, y' + a_0\, y = 0, \qquad t > 0.$$

The proof of this theorem is to verify that the power functions $y(t) = t^{r_\pm}$ solve the differential equation. How did we know we had to try with power functions? One answer could be, this is a guess, a lucky one. Another answer could be that the Euler equation can be transformed into a constant coefficient equation by a change of the independent variable.

Theorem 2.4.4 (Transformation to Constant Coefficients)**.** *The function y is solution of the Euler equidimensional equation*

$$t^2\, y'' + a_1 t\, y' + a_0\, y = 0, \qquad t > 0 \tag{2.4.4}$$

iff the function $u(z) = y(t(z))$, where $t(z) = e^z$, satisfies the constant coefficients equation

$$\ddot{u} + (a_1 - 1)\, \dot{u} + a_0\, u = 0, \qquad z \in \mathbb{R}, \tag{2.4.5}$$

where $y' = dy/dt$ and $\dot{u} = du/dz$.

Remark: The solutions of the constant coefficient equation in (2.4.5) are $u(z) = e^{r_\pm z}$, where $r_\pm$ are the roots of the *characteristic polynomial* of Eq. (2.4.5),

$$r_\pm^2 + (a_1 - 1) r_\pm + a_0 = 0,$$

that is, $r_\pm$ must be a root of the *indicial polynomial* of Eq. (2.4.4).

(a) Consider the case that $r_+ \neq r_-$. Recalling that $y(t) = u(z(t))$, and $z(t) = \ln(t)$, we get

$$y_\pm(t) = u(z(t)) = e^{r_\pm\, z(t)} = e^{r_\pm \ln(t)} = e^{\ln(t^{r_\pm})} \quad \Rightarrow \quad y_\pm(t) = t^{r_\pm}.$$

(b) Consider the case that $r_+ = r_- = r_0$. Recalling that $y(t) = u(z(t))$, and $z(t) = \ln(t)$, we get that $y_+(t) = t^{r_0}$, while the second solution is

$$y_-(t) = u(z(t)) = z(t)\, e^{r_0\, z(t)} = \ln(t)\, e^{r_0 \ln(t)} = \ln(t)\, e^{\ln(t^{r_0})} \quad \Rightarrow \quad y_-(t) = \ln(t)\, t^{r_0}.$$

Proof of Theorem 2.4.4: Given $t > 0$, introduce $t(z) = e^z$. Given a function y, let

$$u(z) = y(t(z)) \quad \Rightarrow \quad u(z) = y(e^z).$$

Then, the derivatives of u and y are related by the chain rule,

$$\dot{u}(z) = \frac{du}{dz}(z) = \frac{dy}{dt}(t(z))\, \frac{dt}{dz}(z) = y'(t(z))\, \frac{d(e^z)}{dz} = y'(t(z))\, e^z$$

so we obtain

$$\dot{u}(z) = t\, y'(t),$$

where we have denoted $\dot{u} = du/dz$. The relation for the second derivatives is

$$\ddot{u}(z) = \frac{d}{dt}\big(t\, y'(t)\big)\, \frac{dt}{dz}(z) = \big(t\, y''(t) + y'(t)\big)\, \frac{d(e^z)}{dz} = \big(t\, y''(t) + y'(t)\big)\, t$$

so we obtain

$$\ddot{u}(z) = t^2\, y''(t) + t\, y'(t).$$

Combining the equations for $\dot{u}$ and $\ddot{u}$ we get

$$t^2\, y'' = \ddot{u} - \dot{u}, \qquad t\, y' = \dot{u}.$$

The function y is solution of the Euler equation $t^2\, y'' + a_1 t\, y' + a_0\, y = 0$ iff holds

$$\ddot{u} - \dot{u} + a_1\, \dot{u} + a_0\, u = 0 \quad \Rightarrow \quad \ddot{u} + (a_1 - 1)\, \dot{u} + a_0\, u = 0.$$

This establishes the Theorem. □

2.4.4. Exercises.

2.4.1.- .

2.4.2.- .

2.5. Nonhomogeneous Equations

All solutions of a linear *homogeneous* equation can be obtained from only two solutions that are linearly independent, called fundamental solutions. Every other solution is a linear combination of these two. This is the general solution formula for homogeneous equations, and it is the main result in § 2.1, Theorem 2.1.7. This result is not longer true for *nonhomogeneous* equations. The superposition property, Theorem 2.1.5, which played an important part to get the general solution formula for homogeneous equations, is not true for nonhomogeneous equations.

We start this section proving a general solution formula for nonhomogeneous equations. We show that all the solutions of the nonhomogeneous equation are a translation by a fixed function of the solutions of the homogeneous equation. The fixed function is one solution—it doesn't matter which one—of the nonhomogenous equation, and it is called a particular solution of the nonhomogeneous equation.

Later in this section we show two different ways to compute the particular solution of a nonhomogeneous equation—the undetermined coefficients method and the variation of parameters method. In the former method we guess a particular solution from the expression of the source in the equation. The guess contains a few unknown constants, the undetermined coefficients, that must be determined by the equation. The undetermined method works for constant coefficients linear operators and simple source functions. The source functions and the associated guessed solutions are collected in a small table. This table is constructed by trial and error. In the latter method we have a formula to compute a particular solution in terms of the equation source, and fundamental solutions of the homogeneous equation. The variation of parameters method works with variable coefficients linear operators and general source functions. But the calculations to find the solution are usually not so simple as in the undetermined coefficients method.

2.5.1. The General Solution Formula.

The general solution formula for homogeneous equations, Theorem 2.1.7, is no longer true for nonhomogeneous equations. But there is a general solution formula for nonhomogeneous equations. Such formula involves three functions, two of them are fundamental solutions of the homogeneous equation, and the third function is any solution of the nonhomogeneous equation. Every other solution of the nonhomogeneous equation can be obtained from these three functions.

Theorem 2.5.1 (General Solution)**.** *Every solution y of the nonhomogeneous equation*

$$L(y) = f, \tag{2.5.1}$$

with $L(y) = y'' + a_1\, y' + a_0\, y$, where a_1, a_0, and f are continuous functions, is given by

$$y = c_1\, y_1 + c_2\, y_2 + y_p,$$

where the functions y_1 and y_2 are fundamental solutions of the homogeneous equation, $L(y_1) = 0$, $L(y_2) = 0$, and y_p is any solution of the nonhomogeneous equation $L(y_p) = f$.

Before we proof Theorem 2.5.1 we state the following definition, which comes naturally from this Theorem.

Definition 2.5.2. *The **general solution** of the nonhomogeneous equation $L(y) = f$ is a two-parameter family of functions*

$$y_{\rm gen}(t) = c_1\, y_1(t) + c_2\, y_2(t) + y_p(t), \tag{2.5.2}$$

where the functions y_1 and y_2 are fundamental solutions of the homogeneous equation, $L(y_1) = 0$, $L(y_2) = 0$, and y_p is any solution of the nonhomogeneous equation $L(y_p) = f$.

Remark: The difference of any two solutions of the nonhomogeneous equation is actually a solution of the homogeneous equation. This is the key idea to prove Theorem 2.5.1. In other words, the solutions of the nonhomogeneous equation are a *translation by a fixed function*, y_p, of the solutions of the homogeneous equation.

Proof of Theorem 2.5.1: Let y be any solution of the nonhomogeneous equation $L(y) = f$. Recall that we already have one solution, y_p, of the nonhomogeneous equation, $L(y_p) = f$. We can now subtract the second equation from the first,

$$L(y) - L(y_p) = f - f = 0 \quad \Rightarrow \quad L(y - y_p) = 0.$$

The equation on the right is obtained from the linearity of the operator L. This last equation says that the difference of any two solutions of the nonhomogeneous equation is solution of the homogeneous equation. The general solution formula for homogeneous equations says that all solutions of the homogeneous equation can be written as linear combinations of a pair of fundamental solutions, y_1, y_2. So the exist constants c_1, c_2 such that

$$y - y_p = c_1 \, y_1 + c_2 \, y_2.$$

Since for every y solution of $L(y) = f$ we can find constants c_1, c_2 such that the equation above holds true, we have found a formula for all solutions of the nonhomogeneous equation. This establishes the Theorem. □

2.5.2. The Undetermined Coefficients Method.

The general solution formula in (2.5.2) is the most useful if there is a way to find a particular solution y_p of the nonhomogeneous equation $L(y_p) = f$. We now present a method to find such particular solution, the undetermined coefficients method. This method works for *linear operators L with constant coefficients* and for *simple source functions f*. Here is a summary of the undetermined coefficients method:

(1) Find fundamental solutions y_1, y_2 of the homogeneous equation $L(y) = 0$.
(2) Given the source functions f, guess the solutions y_p following the Table 1 below.
(3) If the function y_p given by the table satisfies $L(y_p) = 0$, then change the guess to ty_p.. If ty_p satisfies $L(ty_p) = 0$ as well, then change the guess to $t^2 y_p$.
(4) Find the undetermined constants k in the function y_p using the equation $L(y) = f$, where y is y_p, or ty_p or $t^2 y_p$.

$f(t)$ (Source) (K, m, a, b, given.)	$y_p(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
$K_m t^m + \cdots + K_0$	$k_m t^m + \cdots + k_0$
$K_1 \cos(bt) + K_2 \sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$(K_m t^m + \cdots + K_0)\, e^{at}$	$(k_m t^m + \cdots + k_0)\, e^{at}$
$\big(K_1 \cos(bt) + K_2 \sin(bt)\big)\, e^{at}$	$\big(k_1 \cos(bt) + k_2 \sin(bt)\big)\, e^{at}$
$(K_m t^m + \cdots + K_0)\big(\tilde{K}_1 \cos(bt) + \tilde{K}_2 \sin(bt)\big)$	$(k_m t^m + \cdots + k_0)\big(\tilde{k}_1 \cos(bt) + \tilde{k}_2 \sin(bt)\big)$

TABLE 1. List of sources f and solutions y_p to the equation $L(y_p) = f$.

This is the undetermined coefficients method. It is a set of simple rules to find a particular solution y_p of an nonhomogeneous equation $L(y_p) = f$ in the case that the source function f is one of the entries in the Table 1. There are a few formulas in particular cases and a few generalizations of the whole method. We discuss them after a few examples.

Example 2.5.1 (First Guess Right)**.** Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\, e^{2t}.$$

Solution: From the problem we get $L(y) = y'' - 3y' - 4y$ and $f(t) = 3\, e^{2t}$.

(1): Find fundamental solutions y_+, y_- to the homogeneous equation $L(y) = 0$. Since the homogeneous equation has constant coefficients we find the characteristic equation

$$r^2 - 3r - 4 = 0 \quad \Rightarrow \quad r_+ = 4, \quad r_- = -1, \quad \Rightarrow \quad y_+(t) = e^{4t}, \quad y_- = (t) = e^{-t}.$$

(2): The table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = k\, e^{2t}$. The constant k is the undetermined coefficient we must find.

(3): Since $y_p(t) = k\, e^{2t}$ is not solution of the homogeneous equation, we do not need to modify our guess. (Recall: $L(y) = 0$ iff exist constants c_+, c_- such that $y(t) = c_+\, e^{4t} + c_-\, e^{-t}$.)

(4): Introduce y_p into $L(y_p) = f$ and find k. So we do that,

$$(2^2 - 6 - 4)ke^{2t} = 3\, e^{2t} \quad \Rightarrow \quad -6k = 3 \quad \Rightarrow \quad k = -\frac{1}{2}.$$

We guessed that y_p must be proportional to the exponential e^{2t} in order to cancel out the exponentials in the equation above. We have obtained that

$$y_p(t) = -\frac{1}{2}\, e^{2t}.$$

The undetermined coefficients method gives us a way to compute a particular solution y_p of the nonhomogeneous equation. We now use the general solution theorem, Theorem 2.5.1, to write the general solution of the nonhomogeneous equation,

$$y_{\text{gen}}(t) = c_+\, e^{4t} + c_-\, e^{-t} - \frac{1}{2}\, e^{2t}.$$

◁

Remark: The step (4) in Example 2.5.1 is a particular case of the following statement.

Theorem 2.5.3. *Consider the equation $L(y) = f$, where $L(y) = y'' + a_1\, y' + a_0\, y$ has constant coefficients and p is its characteristic polynomial. If the source function is $f(t) = K\, e^{at}$, with $p(a) \neq 0$, then a particular solution of the nonhomogeneous equation is*

$$y_p(t) = \frac{K}{p(a)}\, e^{at}.$$

Proof of Theorem 2.5.3: Since the linear operator L has constant coefficients, let us write L and its associated characteristic polynomial p as follows,

$$L(y) = y'' + a_1 y' + a_0 y, \qquad p(r) = r^2 + a_1 r + a_0.$$

Since the source function is $f(t) = K\, e^{at}$, the Table 1 says that a good guess for a particular soution of the nonhomogneous equation is $y_p(t) = k\, e^{at}$. Our hypothesis is that this guess is not solution of the homogenoeus equation, since

$$L(y_p) = (a^2 + a_1 a + a_0)\, k\, e^{at} = p(a)\, k\, e^{at}, \quad \text{and} \quad p(a) \neq 0.$$

We then compute the constant k using the equation $L(y_p) = f$,

$$(a^2 + a_1 a + a_0)\, k\, e^{at} = K\, e^{at} \quad \Rightarrow \quad p(a)\, k\, e^{at} = K\, e^{at} \quad \Rightarrow \quad k = \frac{K}{p(a)}.$$

We get the particular solution $y_p(t) = \dfrac{K}{p(a)}\, e^{at}$. This establishes the Theorem. □

Remark: As we said, the step (4) in Example 2.5.1 is a particular case of Theorem 2.5.3,

$$y_p(t) = \frac{3}{p(2)}\, e^{2t} = \frac{3}{(2^2 - 6 - 4)}\, e^{2t} = \frac{3}{-6}\, e^{2t} \quad \Rightarrow \quad y_p(t) = -\frac{1}{2}\, e^{2t}.$$

In the following example our first guess for a particular solution y_p happens to be a solution of the homogenous equation.

Example 2.5.2 (First Guess Wrong)**.** Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\, e^{4t}.$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 3\, e^{4t}$, then the operator L is the same as in Example 2.5.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}.$$

The source function is $f(t) = 3\, e^{4t}$, so the Table 1 says that we need to guess $y_p(t) = k\, e^{4t}$. However, this function y_p is solution of the homogeneous equation, because

$$y_p = k\, y_{+} \quad \Rightarrow \quad L(y_p) = 0.$$

We have to change our guess, as indicated in the undetermined coefficients method, step (3)

$$y_p(t) = kt\, e^{4t}.$$

This new guess is not solution of the homogeneous equation. So we proceed to compute the constant k. We introduce the guess into $L(y_p) = f$,

$$y_p' = (1 + 4t)\, k\, e^{4t}, \qquad y_p'' = (8 + 16t)\, k\, e^{4t} \quad \Rightarrow \quad \big[8 - 3 + (16 - 12 - 4)t\big]\, k\, e^{4t} = 3\, e^{4t},$$

therefore, we get that

$$5k = 3 \quad \Rightarrow \quad k = \frac{3}{5} \quad \Rightarrow \quad y_p(t) = \frac{3}{5}\, t\, e^{4t}.$$

The general solution theorem for nonhomogneneous equations says that

$$y_{\text{gen}}(t) = c_{+}\, e^{4t} + c_{-}\, e^{-t} + \frac{3}{5}\, t\, e^{4t}.$$

◁

In the following example the equation source is a trigonometric function.

Example 2.5.3 (First Guess Right)**.** Find all the solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 2\sin(t)$, then the operator L is the same as in Example 2.5.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}.$$

Since the source function is $f(t) = 2\sin(t)$, the Table 1 says that we need to choose the function $y_p(t) = k_1 \cos(t) + k_2 \sin(t)$. This function y_p is not solution to the homogeneous equation. So we look for the constants k_1, k_2 using the differential equation,

$$y_p' = -k_1 \sin(t) + k_2 \cos(t), \qquad y_p'' = -k_1 \cos(t) - k_2 \sin(t),$$

and then we obtain

$$[-k_1 \cos(t) - k_2 \sin(t)] - 3[-k_1 \sin(t) + k_2 \cos(t)] - 4[k_1 \cos(t) + k_2 \sin(t)] = 2\sin(t).$$

Reordering terms in the expression above we get

$$(-5k_1 - 3k_2)\cos(t) + (3k_1 - 5k_2)\sin(t) = 2\sin(t).$$

The last equation must hold for all $t \in \mathbb{R}$. In particular, it must hold for $t = \pi/2$ and for $t = 0$. At these two points we obtain, respectively,

$$\left.\begin{aligned} 3k_1 - 5k_2 &= 2,\\ -5k_1 - 3k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} k_1 = \dfrac{3}{17},\\[2ex] k_2 = -\dfrac{5}{17}. \end{cases}$$

So the particular solution to the nonhomogeneous equation is given by

$$y_p(t) = \frac{1}{17}\big[3\cos(t) - 5\sin(t)\big].$$

The general solution theorem for nonhomogeneous equations implies

$$y_{\text{gen}}(t) = c_+ e^{4t} + c_- e^{-t} + \frac{1}{17}\big[3\cos(t) - 5\sin(t)\big].$$

◁

The next example collects a few nonhomogeneous equations and the guessed function y_p.

Example 2.5.4. We provide few more examples of nonhomogeneous equations and the appropriate guesses for the particular solutions.

(a) For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess, $y_p(t) = \big[k_1 \cos(t) + k_2 \sin(t)\big]\, e^{2t}$.

(b) For $y'' - 3y' - 4y = 2t^2\, e^{3t}$, guess, $y_p(t) = \big(k_2 t^2 + k_1 t + k_0\big)\, e^{3t}$.

(c) For $y'' - 3y' - 4y = 2t^2\, e^{4t}$, guess, $y_p(t) = \big(k_2 t^2 + k_1 t + k_0\big)\, t\, e^{4t}$.

(d) For $y'' - 3y' - 4y = 3t\,\sin(t)$, guess, $y_p(t) = (k_1 t + k_0)\,\big[\tilde{k}_1 \cos(t) + \tilde{k}_2 \sin(t)\big]$.

◁

Remark: Suppose that the source function f does not appear in Table 1, but f can be written as $f = f_1 + f_2$, with f_1 and f_2 in the table. In such case look for a particular solution $y_p = y_{p_1} + y_{p_2}$, where $L(y_{p_1}) = f_1$ and $L(y_{p_2}) = f_2$. Since the operator L is linear,

$$L(y_p) = L(y_{p_1} + y_{p_2}) = L(y_{p_1}) + L(y_{p_2}) = f_1 + f_2 = f \quad \Rightarrow \quad L(y_p) = f.$$

Example 2.5.5. Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\, e^{2t} + 2\sin(t).$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 2\sin(t)$, then the operator L is the same as in Example 2.5.1 and 2.5.3. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in these examples,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}.$$

The source function $f(t) = 3\,e^{2t} + 2\sin(t)$ does not appear in Table 1, but each term does, $f_1(t) = 3\,e^{2t}$ and $f_2(t) = 2\sin(t)$. So we look for a particular solution of the form

$$y_p = y_{p_1} + y_{p_2}, \quad \text{where} \quad L(y_{p_1}) = 3\,e^{2t}, \quad L(y_{p_2}) = 2\sin(t).$$

We have chosen this example because we have solved each one of these equations before, in Example 2.5.1 and 2.5.3. We found the solutions

$$y_{p_1}(t) = -\frac{1}{2}\,e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\,\big(3\cos(t) - 5\sin(t)\big).$$

Therefore, the particular solution for the equation in this example is

$$y_p(t) = -\frac{1}{2}\,e^{2t} + \frac{1}{17}\,\big(3\cos(t) - 5\sin(t)\big).$$

Using the general solution theorem for nonhomogeneous equations we obtain

$$y_{\text{gen}}(t) = c_{+}\,e^{4t} + c_{-}\,e^{-t} - \frac{1}{2}\,e^{2t} + \frac{1}{17}\,\big(3\cos(t) - 5\sin(t)\big).$$

◁

2.5.3. The Variation of Parameters Method.

This method provides a second way to find a particular solution y_p to a nonhomogeneous equation $L(y) = f$. We summarize this method in formula to compute y_p in terms of any pair of fundamental solutions to the homogeneous equation $L(y) = 0$. The variation of parameters method works with second order linear equations having *variable coefficients* and contiuous but otherwise *arbitrary sources*. When the source function of a nonhomogeneous equation is simple enough to appear in Table 1 the undetermined coefficients method is a quick way to find a particular solution to the equation. When the source is more complicated, one usually turns to the variation of parameters method, with its more involved formula for a particular solution.

Theorem 2.5.4 (Variation of Parameters)**.** *A particular solution to the equation*

$$L(y) = f,$$

with $L(y) = y'' + a_1(t)\,y' + a_0(t)\,y$ *and* a_1, a_0, f *continuous functions, is given by*

$$y_p = u_1 y_1 + u_2 y_2,$$

where y_1, y_2 *are fundamental solutions of the homogeneous equatio* $L(y) = 0$ *and the functions* u_1, u_2 *are defined by*

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\,dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\,dt, \tag{2.5.3}$$

where $W_{y_1 y_2}$ *is the Wronskian of* y_1 *and* y_2*.*

The proof is a generalization of the reduction order method. Recall that the reduction order method is a way to find a second solution y_2 of an homogeneous equation if we already know one solution y_1. One writes $y_2 = u\,y_1$ and the original equation $L(y_2) = 0$ provides an equation for u. This equation for u is simpler than the original equation for y_2 because the function y_1 satisfies $L(y_1) = 0$.

The formula for y_p can be seen as a generalization of the reduction order method. We write y_p in terms of both fundamental solutions y_1, y_2 of the homogeneous equation,

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t).$$

We put this y_p in the equation $L(y_p) = f$ and we find an equation relating u_1 and u_2. It is important to realize that we have added one new function to the original problem. The original problem is to find y_p. Now we need to find u_1 and u_2, but we still have only one equation to solve, $L(y_p) = f$. The problem for u_1, u_2 cannot have a unique solution. So we are completely free to add a second equation to the original equation $L(y_p) = f$. We choose the second equation so that we can solve for u_1 and u_2.

Proof of Theorem 2.5.4: Motivated by the reduction of order method we look for a y_p

$$y_p = u_1\, y_1 + u_2\, y_2.$$

We hope that the equations for u_1, u_2 will be simpler to solve than the equation for y_p. But we started with one unknown function and now we have two unknown functions. So we are free to add one more equation to fix u_1, u_2. We choose

$$u_1'\, y_1 + u_2'\, y_2 = 0.$$

In other words, we choose $u_2 = \displaystyle\int -\frac{y_1'}{y_2'}\, u_1'\, dt$. Let's put this y_p into $L(y_p) = f$. We need y_p' (and recall, $u_1'\, y_1 + u_2'\, y_2 = 0$)

$$y_p' = u_1'\, y_1 + u_1\, y_1' + u_2'\, y_2 + u_2\, y_2' \quad \Rightarrow \quad y_p' = u_1\, y_1' + u_2\, y_2'.$$

and we also need y_p'',

$$y_p'' = u_1'\, y_1' + u_1\, y_1'' + u_2'\, y_2' + u_2\, y_2''.$$

So the equation $L(y_p) = f$ is

$$(u_1'\, y_1' + u_1\, y_1'' + u_2'\, y_2' + u_2\, y_2'') + a_1(u_1\, y_1' + u_2\, y_2') + a_0(u_1\, y_1 + u_2\, y_2) = f$$

We reorder a few terms and we see that

$$u_1'\, y_1' + u_2'\, y_2' + u_1\, (y_1'' + a_1\, y_1' + a_0\, y_1) + u_2\, (y_2'' + a_1\, y_2' + a_0\, y_2) = f.$$

The functions y_1 and y_2 are solutions to the homogeneous equation,

$$y_1'' + a_1\, y_1' + a_0\, y_1 = 0, \qquad y_2'' + a_1\, y_2' + a_0\, y_2 = 0,$$

so u_1 and u_2 must be solution of a simpler equation that the one above, given by

$$u_1'\, y_1' + u_2'\, y_2' = f. \tag{2.5.4}$$

So we end with the equations

$$\begin{aligned} u_1'\, y_1' + u_2'\, y_2' &= f \\ u_1'\, y_1 + u_2'\, y_2 &= 0. \end{aligned}$$

And this is a 2×2 algebraic linear system for the unknowns u_1', u_2'. It is hard to overstate the importance of the word "algebraic" in the previous sentence. From the second equation above we compute u_2' and we introduce it in the first equation,

$$u_2' = -\frac{y_1}{y_2}\, u_1' \quad \Rightarrow \quad u_1' y_1' - \frac{y_1 y_2'}{y_2}\, u_1' = f \quad \Rightarrow \quad u_1'\Big(\frac{y_1' y_2 - y_1 y_2'}{y_2}\Big) = f.$$

Recall that the Wronskian of two functions is $W_{12} = y_1 y_2' - y_1' y_2$, we get

$$u_1' = -\frac{y_2 f}{W_{12}} \quad \Rightarrow \quad u_2' = \frac{y_1 f}{W_{12}}.$$

These equations are the derivative of Eq. (2.5.3). Integrate them in the variable t and choose the integration constants to be zero. We get Eq. (2.5.3). This establishes the Theorem. ☐

Remark: The integration constants in the expressions for u_1, u_2 can always be chosen to be zero. To understand the effect of the integration constants in the function y_p, let us do the following. Denote by u_1 and u_2 the functions in Eq. (2.5.3), and given any real numbers c_1 and c_2 define

$$\tilde{u}_1 = u_1 + c_1, \qquad \tilde{u}_2 = u_2 + c_2.$$

Then the corresponding solution $\tilde{y}_p$ is given by

$$\tilde{y}_p = \tilde{u}_1\, y_1 + \tilde{u}_2\, y_2 = u_1\, y_1 + u_2\, y_2 + c_1\, y_1 + c_2\, y_2 \quad \Rightarrow \quad \tilde{y}_p = y_p + c_1\, y_1 + c_2\, y_2.$$

The two solutions $\tilde{y}_p$ and y_p differ by a solution to the homogeneous differential equation. So both functions are also solution to the nonhomogeneous equation. One is then free to choose the constants c_1 and c_2 in any way. We chose them in the proof above to be zero.

Example 2.5.6. Find the general solution of the nonhomogeneous equation

$$y'' - 5y' + 6y = 2\, e^t.$$

Solution: The formula for y_p in Theorem 2.5.4 requires we know fundamental solutions to the homogeneous problem. So we start finding these solutions first. Since the equation has constant coefficients, we compute the characteristic equation,

$$r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(5 \pm \sqrt{25 - 24}\big) \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = 2. \end{cases}$$

So, the functions y_1 and y_2 in Theorem 2.5.4 are in our case given by

$$y_1(t) = e^{3t}, \qquad y_2(t) = e^{2t}.$$

The Wronskian of these two functions is given by

$$W_{y_1 y_2}(t) = (e^{3t})(2\, e^{2t}) - (3\, e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -e^{5t}.$$

We are now ready to compute the functions u_1 and u_2. Notice that Eq. (2.5.3) the following differential equations

$$u_1' = -\frac{y_2 f}{W_{y_1 y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1 y_2}}.$$

So, the equation for u_1 is the following,

$$u_1' = -e^{2t}(2\, e^t)(-e^{-5t}) \quad \Rightarrow \quad u_1' = 2\, e^{-2t} \quad \Rightarrow \quad u_1 = -e^{-2t},$$

$$u_2' = e^{3t}(2\, e^t)(-e^{-5t}) \quad \Rightarrow \quad u_2' = -2\, e^{-t} \quad \Rightarrow \quad u_2 = 2\, e^{-t},$$

where we have chosen the constant of integration to be zero. The particular solution we are looking for is given by

$$y_p = (-e^{-2t})(e^{3t}) + (2\, e^{-t})(e^{2t}) \quad \Rightarrow \quad y_p = e^t.$$

Then, the general solution theorem for nonhomogeneous equation implies

$$y_{\text{gen}}(t) = c_+\, e^{3t} + c_-\, e^{2t} + e^t \qquad c_+, c_- \in \mathbb{R}.$$

◁

Example 2.5.7. Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

Solution: We first rewrite the nonhomogeneous equation above in the form given in Theorem 2.5.4. In this case we must divide the whole equation by t^2,

$$y'' - \frac{2}{t^2}\, y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$

We now proceed to compute the Wronskian of the fundamental solutions y_1, y_2,

$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -3.$$

We now use the equation in (2.5.3) to obtain the functions u_1 and u_2,

$$\begin{aligned} u_1' &= -\frac{1}{t}\Big(3 - \frac{1}{t^2}\Big)\frac{1}{-3} & \qquad u_2' &= (t^2)\Big(3 - \frac{1}{t^2}\Big)\frac{1}{-3} \\ &= \frac{1}{t} - \frac{1}{3}\, t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6} t^{-2}, & &= -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3}\, t^3 + \frac{1}{3} t. \end{aligned}$$

A particular solution to the nonhomogeneous equation above is $\tilde{y}_p = u_1 y_1 + u_2 y_2$, that is,

$$\begin{aligned} \tilde{y}_p &= \Big[\ln(t) + \frac{1}{6} t^{-2}\Big](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1}) \\ &= t^2 \ln(t) + \frac{1}{6} - \frac{1}{3}\, t^2 + \frac{1}{3} \\ &= t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, t^2 \\ &= t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\, y_1(t). \end{aligned}$$

However, a simpler expression for a solution of the nonhomogeneous equation above is

$$y_p = t^2 \ln(t) + \frac{1}{2}.$$

◁

Remark: Sometimes it could be difficult to remember the formulas for functions u_1 and u_2 in (2.5.3). In such case one can always go back to the place in the proof of Theorem 2.5.4 where these formulas come from, the system

$$\begin{aligned} u_1' y_1' + u_2' y_2' &= f \\ u_1' y_1 + u_2' y_2 &= 0. \end{aligned}$$

The system above could be simpler to remember than the equations in (2.5.3). We end this Section using the equations above to solve the problem in Example 2.5.7. Recall that the solutions to the homogeneous equation in Example 2.5.7 are $y_1(t) = t^2$, and $y_2(t) = 1/t$, while the source function is $f(t) = 3 - 1/t^2$. Then, we need to solve the system

$$\begin{aligned} t^2\, u_1' + u_2' \frac{1}{t} &= 0, \\ 2t\, u_1' + u_2' \frac{(-1)}{t^2} &= 3 - \frac{1}{t^2}. \end{aligned}$$

This is an algebraic linear system for u_1' and u_2'. Those are simple to solve. From the equation on top we get u_2' in terms of u_1', and we use that expression on the bottom equation,

$$u_2' = -t^3\, u_1' \quad \Rightarrow \quad 2t\, u_1' + t\, u_1' = 3 - \frac{1}{t^2} \quad \Rightarrow \quad u_1' = \frac{1}{t} - \frac{1}{3t^3}.$$

Substitue back the expression for u_1' in the first equation above and we get u_2'. We get,

$$\begin{aligned} u_1' &= \frac{1}{t} - \frac{1}{3t^3} \\ u_2' &= -t^2 + \frac{1}{3}. \end{aligned}$$

We should now integrate these functions to get u_1 and u_2 and then get the particular solution $\tilde{y}_p = u_1 y_1 + u_2 y_2$. We do not repeat these calculations, since they are done Example 2.5.7.

2.5.4. Exercises.

2.5.1.- .

2.5.2.- .

2.6. Applications

Different physical systems are mathematically identical. In this Section we show that a weight attached to a spring, oscillating either in air or under water, is mathematically identical to the behavior of an electric current in a circuit containing a resistance, a capacitor, and an inductance. Mathematical identical means that both systems are described by the same differential equation.

2.6.1. Review of Constant Coefficient Equations.

In Section 2.3 we have seen how to find solutions to second order, linear, constant coefficient, homogeneous, differential equations,

$$y'' + a_1\, y' + a_0\, y = 0, \qquad a_1, a_2 \in \mathbb{R}. \tag{2.6.1}$$

Theorem 2.3.2 provides formulas for the general solution of this equation. We review here this result, and at the same time we introduce new names describing these solutions, names that are common in the physics literature. The first step to obtain solutions to Eq. (2.6.1) is to find the roots or the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, which are given by

$$r_\pm = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

We then have three different cases to consider.

(a) A system is *over damped* in the case that $a_1^2 - 4a_0 > 0$. In this case the characteristic polynomial has real and distinct roots, r_+, r_-, and the corresponding solutions to the differential equation are

$$y_1(t) = e^{r_+ t}, \qquad y_2(t) = e^{r_- t}.$$

So the solutions are exponentials, increasing or decreasing, according whether the roots are positive or negative, respectively. The decreasing exponential solutions originate the name over damped solutions.

(b) A system is *critically damped* in the case that $a_1^2 - 4a_0 = 0$. In this case the characteristic polynomial has only one real, repeated, root, $\hat{r} = -a_1/2$, and the corresponding solutions to the differential equation are then,

$$y_1(t) = e^{-a_1 t/2}, \qquad y_2(t) = t\, e^{-a_1 t/2}.$$

(c) A system is *under damped* in the case that $a_1^2 - 4a_0 < 0$. In this case the characteristic polynomial has two complex roots, $r_\pm = \alpha \pm \beta i$, where one root is the complex conjugate of the other, since the polynomial has real coefficients. The corresponding solutions to the differential equation are

$$y_1(t) = e^{\alpha t}\cos(\beta t), \qquad y_2(t) = e^{\alpha t}\sin(\beta t).$$

where $\alpha = -\dfrac{a_1}{2}$ and $\beta = \dfrac{1}{2}\sqrt{4a_0 - a_1^2}$.

(d) A system is *undamped* when is under damped with $a_1 = 0$. Therefore, the characteristic polynomial has two pure imaginary roots $r_\pm = \pm\sqrt{a_0}$. The corresponding solutions are oscillatory functions,

$$y_1(t) = \cos(\omega_0 t), \qquad y_2(t) = \sin(\omega_0 t).$$

where $\omega_0 = \sqrt{a_0}$.

2.6.2. Undamped Mechanical Oscillations.

Springs are curious objects, when you slightly deform them they create a force proportional and in opposite direction to the deformation. When you release the spring, it goes back to its original size. This is true for small enough deformations. If you stretch the spring long enough, the deformations are permanent.

Definition 2.6.1. *A **spring** is an object that when deformed by an amount Δl creates a force $F_s = -k\,\Delta l$, with $k > 0$.*

Consider a spring-body system as shown in Fig. 2.6.2. A spring is fixed to a ceiling and hangs vertically with a natural length l. It stretches by Δl when a body with mass m is attached to its lower end, just as in the middle spring in Fig. 2.6.2. We assume that the weight m is small enough so that the spring is not damaged. This means that the spring acts like a normal spring, whenever it is deformed by an amount Δl it makes a force proportional and opposite to the deformation,

$$F_{s0} = -k\,\Delta l.$$

Here $k > 0$ is a constant that depends on the type of spring. Newton's law of motion imply the following result.

Theorem 2.6.2. *A spring-body system with spring constant k, body mass m, at rest with a spring deformation Δl, within the rage where the spring acts like a spring, satisfies*

$$mg = k\,\Delta l.$$

Proof of Theorem 2.6.2: Since the spring-body system is at rest, Newton's law of motion imply that all forces acting on the body must add up to zero. The only two forces acting on the body are its weight, $F_g = mg$, and the force done by the spring, $F_{s0} = -k\,\Delta l$. We have used the hypothesis that Δl is small enough so the spring is not damaged. We are using the sign convention displayed in Fig. 2.6.2, where forces pointing downwards are positive.

As we said above, since the body is at rest, the addition of all forces acting on the body must vanish,

$$F_g + F_{s0} = 0 \quad \Rightarrow \quad mg = k\,\Delta l.$$

This establishes the Theorem. □

Remark: Rewriting the equation above as

$$k = \frac{mg}{\Delta l}.$$

it is possible to compute the spring constant k by measuring the displacement Δl and knowing the body mass m.

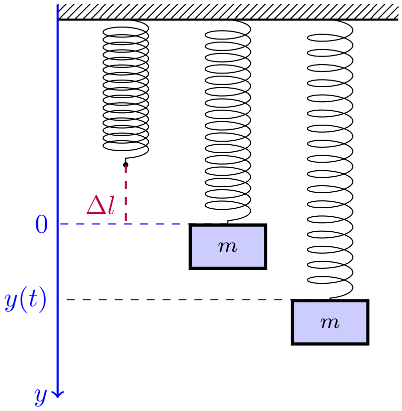


FIGURE 2. Springs with weights.

We now find out how the body will move when we take it away from the rest position. To describe that movement we introduce a vertical coordinate for the displacements, y, as shown in Fig. 2.6.2, with y positive downwards, and $y = 0$ at the rest position of the spring and the body. The physical system we want to describe is simple; we further stretch the spring with the body by y_0 and then we release it with an initial velocity v_0. Newton's law of motion determine the subsequent motion.

Theorem 2.6.3. *The vertical movement of a spring-body system in air with spring constant $k > 0$ and body mass $m > 0$ is described by the solutions of the differential equation*

$$m\,y'' + k\,y = 0, \tag{2.6.2}$$

where y is the vertical displacement function as shown in Fig. 2.6.2. Furthermore, there is a unique solution to Eq. (2.6.2) satisfying the initial conditions $y(0) = y_0$ and $y'(0) = v_0$,

$$y(t) = A\,\cos(\omega_0 t - \phi),$$

with angular frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$, where the amplitude $A \geqslant 0$ and phase-shift $\phi \in (-\pi, \pi]$,

$$A = \sqrt{y_0^2 + \frac{v_0^2}{\omega_0^2}}, \qquad \phi = \arctan\Big(\frac{v_0}{\omega_0 y_0}\Big).$$

Remark: The angular or circular frequency of the system is $\omega_0 = \sqrt{k/m}$, meaning that the motion of the system is periodic with *period* given by $T = 2\pi/\omega_0$, which in turns implies that the system *frequency* is $\nu_0 = \omega_0/(2\pi)$.

Proof of Theorem 2.6.3: Newton's second law of motion says that mass times acceleration of the body $m\,y''(t)$ must be equal to the sum of all forces acting on the body, hence

$$m\,y''(t) = F_g + F_{s0} + F_s(t),$$

where $F_s(t) = -k\,y(t)$ is the force done by the spring due to the extra displacement y. Since the first two terms on the right hand side above cancel out, $F_g + F_{s0} = 0$, the body displacement from the equilibrium position, $y(t)$, must be solution of the differential equation

$$m\,y''(t) + k\,y(t) = 0.$$

which is Eq. (2.6.2). In Section **??** we have seen how to solve this type of differential equations. The characteristic polynomial is $p(r) = mr^2 + k$, which has complex roots $r_\pm = \pm\omega_0^2\,i$, where we introduced the angular or circular frequency of the system,

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

The reason for this name is the calculations done in Section **??**, where we found that a real-valued expression for the general solution to Eq. (2.6.2) is given by

$$y_{\rm gen}(t) = c_1\,\cos(\omega_0 t) + c_2\,\sin(\omega_0 t).$$

This means that the body attached to the spring oscillates around the equilibrium position $y = 0$ with period $T = 2\pi/\omega_0$, hence frequency $\nu_0 = \omega_0/(2\pi)$. There is an equivalent way to express the general solution above given by

$$y_{\rm gen}(t) = A\,\cos(\omega_0 t - \phi).$$

These two expressions for $y_{\rm gen}$ are equivalent because of the trigonometric identity

$$A\cos(\omega_0 t - \phi) = A\cos(\omega_0 t)\cos(\phi) + A\sin(\omega_0 t)\sin(\phi),$$

which holds for all A and ϕ, and $\omega_0 t$. Then, it is not difficult to see that

$$\left.\begin{aligned} c_1 &= A\cos(\phi),\\ c_2 &= A\sin(\phi). \end{aligned}\right\} \quad\Leftrightarrow\quad \left\{\begin{aligned} A &= \sqrt{c_1^2 + c_2^2},\\ \phi &= \arctan\Big(\frac{c_2}{c_1}\Big). \end{aligned}\right.$$

Since both expressions for the general solution are equivalent, we use the second one, in terms of the amplitude and phase-shift. The initial conditions $y(0) = y_0$ and $y'(0) = \hat{y}_0$ determine the constants A and ϕ. Indeed,

$$\left.\begin{aligned} y_0 &= y(0) = A\,\cos(\phi),\\ v_0 &= y'(0) = A\omega_0\,\sin(\phi). \end{aligned}\right\} \quad\Rightarrow\quad \begin{cases} A = \sqrt{y_0^2 + \dfrac{v_0^2}{\omega_0^2}},\\[2ex] \phi = \arctan\Bigl(\dfrac{v_0}{\omega_0 y_0}\Bigr). \end{cases}$$

This establishes the Theorem. $\square$

Example 2.6.1. Find the movement of a 50 gr mass attached to a spring moving in air with initial conditions $y(0) = 4\,\text{cm}$ and $y'(0) = 40\,\text{cm/s}$. The spring is such that a 30 gr mass stretches it 6 cm. Approximate the acceleration of gravity by $1000\,\text{cm/s}^2$.

Solution: Theorem 2.6.3 says that the equation satisfied by the displacement y is given by

$$my'' + ky = 0.$$

In order to solve this equation we need to find the spring constant, k, which by Theorem 2.6.2 is given by $k = mg/\Delta l$. In our case when a mass of $m = 30\,\text{gr}$ is attached to the sprint, it stretches $\Delta l = 6\,\text{cm}$, so we get,

$$k = \frac{(30)\,(1000)}{6} \quad\Rightarrow\quad k = 5000\,\frac{\text{gr}}{\text{s}^2}.$$

Knowing the spring constant k we can now describe the movement of the body with mass $m = 50\,\text{gr}$. The solution of the differential equation above is obtained as usual, first find the roots of the characteristic polynomial

$$mr^2 + k = 0 \quad\Rightarrow\quad r_\pm = \pm\omega_0 i, \quad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{50}} \quad\Rightarrow\quad \omega_0 = 10\,\frac{1}{\text{s}}.$$

We write down the general solution in terms of the amplitude A and phase-shift ϕ,

$$y(t) = A\,\cos(\omega_0 t - \phi) \quad\Rightarrow\quad y(t) = A\,\cos(10\,t - \phi).$$

To accommodate the initial conditions we need the function $y'(t) = -A\omega_0\,\sin(\omega_0 t - \phi)$. The initial conditions determine the amplitude and phase-shift, as follows,

$$\left.\begin{aligned} 4 &= y(0) = A\cos(\phi),\\ 40 &= y'(0) = -10\,A\sin(-\phi) \end{aligned}\right\} \quad\Rightarrow\quad \begin{cases} A = \sqrt{16+16},\\[1ex] \phi = \arctan\Bigl(\dfrac{40}{(10)(4)}\Bigr). \end{cases}$$

We obtain that $A = 4\sqrt{2}$ and $\tan(\phi) = 1$. The later equation implies that either $\phi = \pi/4$ or $\phi = -3\pi/4$, for $\phi \in (-\pi, \pi]$. If we pick the second value, $\phi = -3\pi/4$, this would imply that $y(0) < 0$ and $y'(0) < 0$, which is not true in our case. So we **must pick** the value $\phi = \pi/4$. We then conclude:

$$y(t) = 4\sqrt{2}\,\cos\Bigl(10\,t - \frac{\pi}{4}\Bigr).$$

$\triangleleft$

2.6.3. Damped Mechanical Oscillations.

Suppose now that the body in the spring-body system is a thin square sheet of metal. If the main surface of the sheet is perpendicular to the direction of motion, then the air dragged by the sheet during the spring oscillations will be significant enough to slow down the spring oscillations in an appreciable time. One can find out that the friction force done by the air opposes the movement and it is proportional to the velocity of the body, that is, $F_d = -d\,y'(t)$. We call such force a *damping force*, where

$d > 0$ is the damping coefficient, and systems having such force damped systems. We now describe the spring-body system in the case that there is a non-zero damping force.

Theorem 2.6.4.

(a) The vertical displacement y, function as shown in Fig. 2.6.2, of a spring-body system with spring constant $k > 0$, body mass $m > 0$, and damping constant $d \geqslant 0$, is described by the solutions of

$$m\,y'' + d\,y' + k\,y = 0, \tag{2.6.3}$$

(b) The roots of the characteristic polynomial of Eq. (2.6.3) are $r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}$, with damping coefficient $\omega_d = \dfrac{d}{2m}$ and circular frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$.

(c) The solutions to Eq. (2.6.3) fall into one of the following cases:

(i) A system with $\omega_d > \omega_0$ is called over damped, with general solution to Eq. (2.6.3)

$$y(t) = c_+\, e^{r_+ t} + c_-\, e^{r_- t}.$$

(ii) A system with $\omega_d = \omega_0$ is called critically damped, with general solution to Eq. (2.6.3)

$$y(t) = c_+\, e^{-\omega_d t} + c_-\, t\, e^{-\omega_d t}.$$

(iii) A system with $\omega_d < \omega_0$ is called under damped, with general solution to Eq. (2.6.3)

$$y(t) = A\, e^{-\omega_d t}\, \cos(\beta t - \phi),$$

where $\beta = \sqrt{\omega_0^2 - \omega_d^2}$.

(d) There is a unique solution to Eq. (2.6.2) with initial conditions $y(0) = y_0$ and $y'(0) = v_0$.

Remark: In the case the damping coefficient vanishes we recover Theorem 2.6.3.

Proof of Therorem 2.6.3: Newton's second law of motion says that mass times acceleration of the body $m\,y''(t)$ must be equal to the sum of all forces acting on the body. In the case that we take into account the air dragging force we have

$$m\,y''(t) = F_g + F_{s0} + F_s(t) + F_d(t),$$

where $F_s(t) = -k\,y(t)$ as in Theorem 2.6.3, and $F_d(t) = -d\,y'(t)$ is the air -body dragging force. Since the first two terms on the right hand side above cancel out, $F_g + F_{s0} = 0$, as mentioned in Theorem 2.6.2, the body displacement from the equilibrium position, $y(t)$, must be solution of the differential equation

$$m\,y''(t) + +d\,y'(t) + k\,y(t) = 0.$$

which is Eq. (2.6.3). In Section **??** we have seen how to solve this type of differential equations. The characteristic polynomial is $p(r) = mr^2 + dr + k$, which has complex roots

$$r_\pm = \frac{1}{2m}\big[-d \pm \sqrt{d^2 - 4mk}\big] = -\frac{d}{2m} \pm \sqrt{\Big(\frac{d}{2m}\Big)^2 - \frac{k}{m}} \quad\Rightarrow\quad r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}.$$

where $\omega_d = \dfrac{d}{2m}$ and $\omega_0 = \sqrt{\dfrac{k}{m}}$. In Section **??** we found that the general solution of a differential equation with a characteristic polynomial having roots as above can be divided into three groups. For the case $r_+ \neq r_-$ real valued, we obtain case (ci), for the case $r_+ = r_-$ we obtain case (cii). Finally, we said that the general solution for the case of two complex roots $r_\pm = \alpha + \beta i$ was given by

$$y_(t) = e^{\alpha t}\big(c_1 \cos(\beta t) + c_2 \sin(\beta t)\big).$$

In our case $\alpha = -\omega_d$ and $\beta = \sqrt{\omega_0^2 - \omega_d^2}$. We now rewrite the second factor on the right-hand side above in terms of an amplitude and a phase shift,

$$y(t) = A\, e^{-\omega_d t} \cos(\beta t - \phi).$$

The main result from Section **??** says that the initial value problem in Theorem 2.6.4 has a unique solution for each of the three cases above. This establishes the Theorem. $\square$

Example 2.6.2. Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a mediuith damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: By Theorem 2.6.4 the differential equation for this system is $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$. The roots of the characteristic polynomial are

$$r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m} = \frac{1}{2}, \quad \omega_0 = \sqrt{\frac{k}{m}} = 1,$$

that is,

$$r_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

This means our system has under damped oscillations. Following Theorem 2.6.4 part (ciii), the general solution is given by

$$y(t) = A\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big).$$

We only need to introduce the initial conditions into the expression for y to find out the amplitude A and phase-shift ϕ. In order to do that we first compute the derivative,

$$y'(t) = -\frac{1}{2}\, A\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big) - \frac{\sqrt{3}}{2}\, A\, e^{-t/2} \sin\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big).$$

The initial conditions in the example imply,

$$\sqrt{3} = y(0) = A\cos(\phi), \qquad 0 = y'(0) = -\frac{1}{2}\, A\cos(\phi) + \frac{\sqrt{3}}{2}\, A\sin(\phi).$$

The second equation above allows us to compute the phase-shift, since

$$\tan(\phi) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \phi = \frac{\pi}{6}, \quad \text{or} \quad \phi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}.$$

If $\phi = -5\pi/6$, then $y(0) < 0$, which is not out case. Hence we **must choose** $\phi = \pi/6$. With that phase-shift, the amplitude is given by

$$\sqrt{3} = A\cos\Big(\frac{\pi}{6}\Big) = A\,\frac{\sqrt{3}}{2} \quad \Rightarrow \quad A = 2.$$

We conclude: $y(t) = 2\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \frac{\pi}{6}\Big)$. $\lhd$

2.6.4. Electrical Oscillations.

We describe the electric current flowing through an RLC-series electric circuit, which consists of a resistance, a coil, and a capacitor connected in series as shown in Fig. 3. A current can be started by approximating a magnet to the coil. If the circuit has low resistance, the current will keep flowing through the coil between the capacitor plates, endlessly. There is no need of a power source to keep the current flowing. The presence of a resistance transforms the current energy into heat, damping the current oscillation.

This system is described by an integro-differential equation found by Kirchhoff, now called Kirchhoff's voltage law, relating the *resistor* R, *capacitor* C, *inductor* L, and the current I in a circuit as follows,

$$L\,I'(t) + R\,I(t) + \frac{1}{C}\int_{t_0}^{t} I(s)\,ds = 0. \quad (2.6.4)$$

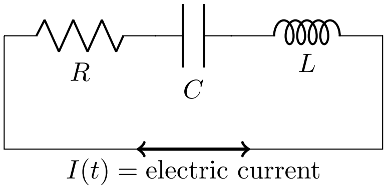


FIGURE 3. An RLC circuit.

Kirchhoff's voltage law is all we need to present the following result.

Theorem 2.6.5. *The electric current I in an RLC circuit with resistance $R \geqslant 0$, capacitance $C > 0$, and inductance $L > 0$, satisfies the differential equation*

$$L\,I''(t) + R\,I'(t) + \frac{1}{C}\,I(t) = 0.$$

The roots of the characteristic polynomial of Eq. (2.6.3) are $r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}$, with damping coefficient $\omega_d = \dfrac{R}{2L}$ and circular frequency $\omega_0 = \sqrt{\dfrac{1}{LC}}$. Furthermore, the results in Theorem 2.6.4 parts (c), (d), hold with ω_d and ω_0 defined here.

Proof of Theorem 2.6.5: Compute the derivate on both sides in Eq. (2.6.4),

$$L\,I''(t) + R\,I'(t) + \frac{1}{C}\,I(t) = 0,$$

and divide by L,

$$I''(t) + 2\Big(\frac{R}{2L}\Big)\,I'(t) + \frac{1}{LC}\,I(t) = 0.$$

Introduce $\omega_d = \dfrac{R}{2L}$ and $\omega_0 = \dfrac{1}{\sqrt{LC}}$, then Kirchhoff's law can be expressed as the second order, homogeneous, constant coefficients, differential equation

$$I'' + 2\omega_d\,I' + \omega_0^2\,I = 0.$$

The rest of the proof follows the one of Theorem 2.6.4. This establishes the Theorem. □

Example 2.6.3. Find real-valued fundamental solutions to $I'' + 2\omega_d\,I' + \omega_0^2\,I = 0$, where $\omega_d = R/(2L)$, $\omega_0^2 = 1/(LC)$, in the cases (a), (b) below.

Solution: The roots of the characteristic polynomial, $p(r) = r^2 + 2\omega_d r + \omega_0^2$, are given by

$$r_\pm = \frac{1}{2}\big[-2\omega_d \pm \sqrt{4\omega_d^2 - 4\omega_0^2}\big] \quad \Rightarrow \quad r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}.$$

Case (a): $R = 0$. This implies $\omega_d = 0$, so $r_\pm = \pm i\omega_0$. Therefore,

$$I_1(t) = \cos(\omega_0 t), \qquad I_2(t) = \sin(\omega_0 t).$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.

Case (b): $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \;\Leftrightarrow\; \frac{R^2}{4L^2} < \frac{1}{LC} \;\Leftrightarrow\; \omega_d^2 < \omega_0^2.$$

Therefore, the characteristic polynomial has complex roots $r_{\pm} = -\omega_d \pm i\sqrt{\omega_0^2 - \omega_d^2}$, hence the fundamental solutions are

$$
\begin{aligned}
I_1(t) &= e^{-\omega_d t}\,\cos(\beta\,t),\\
I_2(t) &= e^{-\omega_d t}\,\sin(\beta\,t),
\end{aligned}
$$

with $\beta = \sqrt{\omega_0^2 - \omega_d^2}$. Therefore, the resistance R damps the current oscillations produced by the capacitor and the inductance. ◁

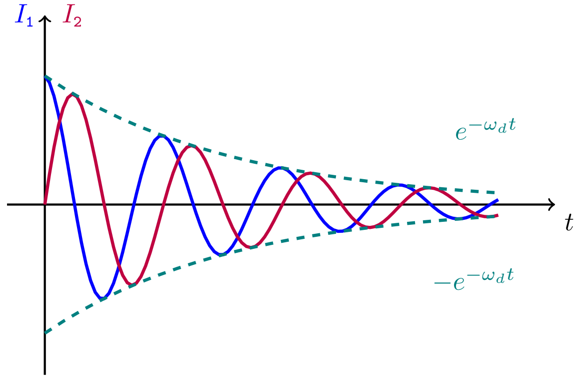


Figure 4. Typical currents I_1, I_2 for case (b).

2.6.5. Exercises.

2.6.1.- .

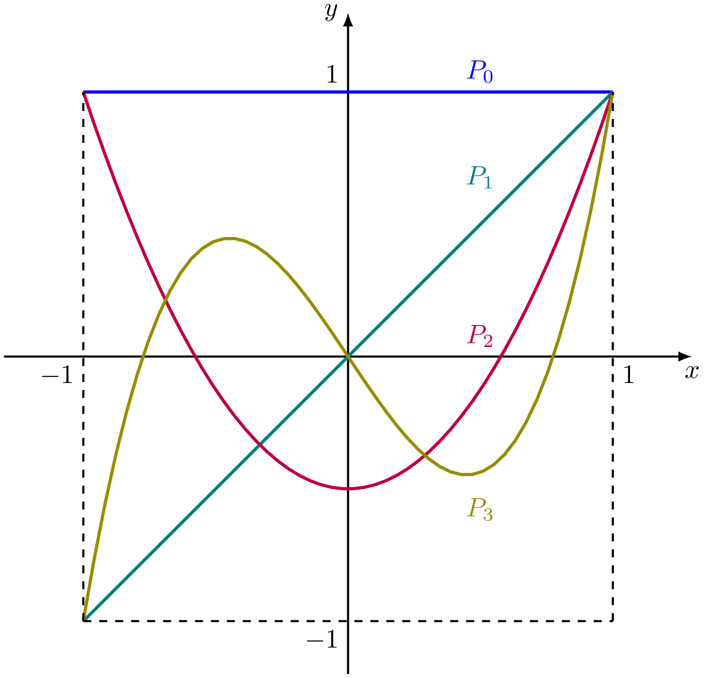
2.6.2.- .

CHAPTER 3

Power Series Solutions

The first differential equations were solved around the end of the seventeen century and beginning of the eighteen century. We studied a few of these equations in § 1.1-1.4 and the constant coefficients equations in Chapter 2. By the middle of the eighteen century people realized that the methods we learnt in these first sections had reached a dead end. One reason was the lack of functions to write the solutions of differential equations. The elementary functions we use in calculus, such as polynomials, quotient of polynomials, trigonometric functions, exponentials, and logarithms, were simply not enough. People even started to think of differential equations as sources to find new functions. It was matter of little time before mathematicians started to use power series expansions to find solutions of differential equations. Convergent power series define functions far more general than the elementary functions from calculus.

In § 3.1 we study the simplest case, when the power series is centered at a regular point of the equation. The coefficients of the equation are analytic functions at regular points, in particular continuous. In § **??** we study the Euler equidimensional equation. The coefficients of an Euler equation diverge at a particular point in a very specific way. No power series are needed to find solutions in this case. In § 3.2 we solve equations with regular singular points. The equation coefficients diverge at regular singular points in a way similar to the coefficients in an Euler equation. We will find solutions to these equations using the solutions to an Euler equation and power series centered precisely at the regular singular points of the equation.


y
1
P_0
P_1
P_2
-1
1
x
P_3
-1

3.1. Solutions Near Regular Points

We study second order linear homogeneous differential equations with variable coefficients,

$$y'' + p(x)\, y' + q(x)\, y = 0.$$

We look for solutions on a domain where the equation coefficients p, q are analytic functions. Recall that a function is analytic on a given domain iff it can be written as a convergent power series expansions on that domain. In Appendix B we review a few ideas on analytic functions and power series expansion that we need in this section. A regular point of the equation is every point where the equation coefficients are analytic. We look for solutions that can be written as power series centered at a regular point. For simplicity we solve only homogeneous equations, but the power series method can be used with nonhomogeneous equations without introducing substantial modifications.

3.1.1. Regular Points.

We now look for solutions to second order linear homogeneous differential equations having variable coefficients. Recall we solved the constant coefficient case in Chapter 2. We have seen that the solutions to constant coefficient equations can be written in terms of elementary functions such as quotient of polynomials, trigonometric functions, exponentials, and logarithms. For example, the equation

$$y'' + y = 0$$

has the fundamental solutions $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$. But the equation

$$x\, y'' + y' + x\, y = 0$$

cannot be solved in terms of elementary functions, that is in terms of quotients of polynomials, trigonometric functions, exponentials and logarithms. Except for equations with constant coefficient and equations with variable coefficient that can be transformed into constant coefficient by a change of variable, no other second order linear equation can be solved in terms of elementary functions. Still, we are interested in finding solutions to variable coefficient equations. Mainly because these equations appear in the description of so many physical systems.

We have said that power series define more general functions than the elementary functions mentioned above. So we look for solutions using power series. In this section we center the power series at a regular point of the equation.

Definition 3.1.1. *A point $x_0 \in \mathbb{R}$ is called a* ***regular point*** *of the equation*

$$y'' + p(x)\, y' + q(x)\, y = 0, \tag{3.1.1}$$

iff p, q are analytic functions at x_0. Otherwise x_0 is called a ***singular point*** *of the equation.*

Remark: Near a regular point x_0 the coefficients p and q in the differential equation above can be written in terms of power series centered at x_0,

$$p(x) = p_0 + p_1\,(x - x_0) + p_2\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} p_n\,(x - x_0)^n,$$

$$q(x) = q_0 + q_1\,(x - x_0) + q_2\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} q_n\,(x - x_0)^n,$$

and these power series converge in a neighborhood of x_0.

Example 3.1.1. Find all the regular points of the equation

$$x\, y'' + y' + x^2\, y = 0.$$

Solution: We write the equation in the form of Eq. (3.1.1),

$$y'' + \frac{1}{x}\, y' + x\, y = 0.$$

In this case the coefficient functions are $p(x) = 1/x$, and $q(x) = x$. The function q is analytic in $\mathbb{R}$. The function p is analytic for all points in $\mathbb{R} - \{0\}$. So the point $x_0 = 0$ is a singular point of the equation. Every other point is a regular point of the equation. ◁

3.1.2. The Power Series Method. The differential equation in (3.1.1) is a particular case of the equations studied in § 2.1, and the existence result in Theorem 2.1.2 applies to Eq. (3.1.1). This Theorem was known to Lazarus Fuchs, who in 1866 added the following: If the coefficient functions p and q are analytic on a domain, so is the solution on that domain. Fuchs went ahead and studied the case where the coefficients p and q have singular points, which we study in § 3.2. The result for analytic coefficients is summarized below.

Theorem 3.1.2. *If the functions p, q are analytic on an open interval $(x_0 - \rho, x_0 + \rho) \subset \mathbb{R}$, then the differential equation*

$$y'' + p(x)\, y' + q(x)\, y = 0,$$

has two independent solutions, y_1, y_2, which are analytic on the same interval.

Remark: A complete proof of this theorem can be found in [2], Page 169. See also [10], § 29. We present the first steps of the proof and we leave the convergence issues to the latter references. The proof we present is based on power series expansions for the coefficients p, q, and the solution y. This is not the proof given by Fuchs in 1866.

Proof of Thorem 3.1.2: Since the coefficient functions p and q are analytic functions on $(x_0 - \rho, x_0 + \rho)$, where $\rho > 0$, they can be written as power series centered at x_0,

$$p(x) = \sum_{n=0}^{\infty} p_n\,(x - x_0)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n\,(x - x_0)^n.$$

We look for solutions that can also be written as power series expansions centered at x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n.$$

We start computing the first derivatives of the function y,

$$y'(x) = \sum_{n=0}^{\infty} n a_n\,(x - x_0)^{(n-1)} \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n\,(x - x_0)^{(n-1)},$$

where in the second expression we started the sum at $n = 1$, since the term with $n = 0$ vanishes. Relabel the sum with $m = n - 1$, so when $n = 1$ we have that $m = 0$, and $n = m + 1$. Therefore, we get

$$y'(x) = \sum_{m=0}^{\infty} (m+1) a_{(m+1)}\,(x - x_0)^m.$$

We finally rename the summation index back to n,

$$y'(x) = \sum_{n=0}^{\infty} (n+1) a_{(n+1)}\,(x - x_0)^n. \tag{3.1.2}$$

From now on we do these steps at once, and the notation $n-1=m \to n$ means

$$y'(x) = \sum_{n=1}^{\infty} n a_n \,(x-x_0)^{(n-1)} = \sum_{n=0}^{\infty} (n+1) a_{(n+1)} \,(x-x_0)^n.$$

We continue computing the second derivative of function y,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n \,(x-x_0)^{(n-2)},$$

and the transformation $n-2=m \to n$ gives us the expression

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} \,(x-x_0)^n.$$

The idea now is to put all these power series back in the differential equation. We start with the term

$$\begin{aligned} q(x)\,y &= \Big(\sum_{n=0}^{\infty} q_n (x-x_0)^n\Big) \Big(\sum_{m=0}^{\infty} a_m (x-x_0)^m\Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k q_{n-k}\Big) \,(x-x_0)^n, \end{aligned}$$

where the second expression above comes from standard results in power series multiplication. A similar calculation gives

$$\begin{aligned} p(x)\,y' &= \Big(\sum_{n=0}^{\infty} p_n (x-x_0)^n\Big) \Big(\sum_{m=0}^{\infty} (m+1) a_{(m+1)} (x-x_0)^m\Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} (k+1) a_{(k+1)} p_{n-k}\Big) \,(x-x_0)^n. \end{aligned}$$

Therefore, the differential equation $y'' + p(x)\,y' + q(x)\,y = 0$ has now the form

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1) a_{(n+2)} + \sum_{k=0}^{n} \big[(k+1) a_{(k+1)} p_{(n-k)} + a_k q_{(n-k)}\big]\Big] \,(x-x_0)^n = 0.$$

So we obtain a *recurrence relation* for the coefficients a_n,

$$(n+2)(n+1) a_{(n+2)} + \sum_{k=0}^{n} \big[(k+1) a_{(k+1)} p_{(n-k)} + a_k q_{(n-k)}\big] = 0,$$

for $n = 0, 1, 2, \cdots$. Equivalently,

$$a_{(n+2)} = -\frac{1}{(n+2)(n+1)} \sum_{k=0}^{n} \big[(k+1) a_{(k+1)} p_{(n-k)} + a_k q_{(n-k)}. \tag{3.1.3}$$

We have obtained an expression for $a_{(n+2)}$ in terms of the previous coefficients $a_{(n+1)}, \cdots, a_0$ and the coefficients of the function p and q. If we choose arbitrary values for the first two coefficients a_0 and a_1, the the recurrence relation in (3.1.3) define the remaining coefficients $a_2, a_3, \cdots$ in terms of a_0 and a_1. The coefficients a_n chosen in such a way guarantee that the function y defined in (3.1.2) satisfies the differential equation.

In order to finish the proof of Theorem 3.1.2 we need to show that the power series for y defined by the recurrence relation actually converges on a nonempty domain, and furthermore that this domain is the same where p and q are analytic. This part of the proof is too complicated for us. The interested reader can find the rest of the proof in [2], Page 169. See also [10], § 29. □

It is important to understand the main ideas in the proof above, because we will follow these ideas to find power series solutions to differential equations. So we now summarize the main steps in the proof above:

(a) Write a power series expansion of the solution centered at a regular point x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^n.$$

(b) Introduce the power series expansion above into the differential equation and find a *recurrence relation* among the coefficients a_n.

(c) Solve the recurrence relation in terms of free coefficients.

(d) If possible, add up the resulting power series for the solutions y_1, y_2.

We follow these steps in the examples below to find solutions to several differential equations. We start with a first order constant coefficient equation, and then we continue with a second order constant coefficient equation. The last two examples consider variable coefficient equations.

Example 3.1.2. Find a power series solution y around the point $x_0 = 0$ of the equation

$$y' + c\,y = 0, \qquad c \in \mathbb{R}.$$

Solution: We already know every solution to this equation. This is a first order, linear, differential equation, so using the method of integrating factor we find that the solution is

$$y(x) = a_0 \, e^{-c\,x}, \qquad a_0 \in \mathbb{R}.$$

We are now interested in obtaining such solution with the power series method. Although this is not a second order equation, the power series method still works in this example. Propose a solution of the form

$$y = \sum_{n=0}^{\infty} a_n \, x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n \, x^{(n-1)}.$$

We can start the sum in y' at $n = 0$ or $n = 1$. We choose $n = 1$, since it is more convenient later on. Introduce the expressions above into the differential equation,

$$\sum_{n=1}^{\infty} n a_n \, x^{n-1} + c \sum_{n=0}^{\infty} a_n \, x^n = 0.$$

Relabel the first sum above so that the functions x^{n-1} and x^n in the first and second sum have the same label. One way is the following,

$$\sum_{n=0}^{\infty} (n+1) a_{(n+1)} \, x^n + \sum_{n=0}^{\infty} c \, a_n \, x^n = 0$$

We can now write down both sums into one single sum,

$$\sum_{n=0}^{\infty} \big[(n+1) a_{(n+1)} + c \, a_n \big] \, x^n = 0.$$

Since the function on the left-hand side must be zero for every $x \in \mathbb{R}$, we conclude that every coefficient that multiplies x^n must vanish, that is,

$$(n+1) a_{(n+1)} + c \, a_n = 0, \qquad n \geqslant 0.$$

The last equation is called a *recurrence relation* among the coefficients a_n. The solution of this relation can be found by writing down the first few cases and then guessing the general expression for the solution, that is,

$$\begin{aligned}
n&=0, & a_1 &= -c\,a_0 & &\Rightarrow & a_1 &= -c\,a_0,\\
n&=1, & 2a_2 &= -c\,a_1 & &\Rightarrow & a_2 &= \frac{c^2}{2!}\,a_0,\\
n&=2, & 3a_3 &= -c\,a_2 & &\Rightarrow & a_3 &= -\frac{c^3}{3!}\,a_0,\\
n&=3, & 4a_4 &= -c\,a_3 & &\Rightarrow & a_4 &= \frac{c^4}{4!}\,a_0.
\end{aligned}$$

One can check that the coefficient a_n can be written as

$$a_n = (-1)^n \frac{c^n}{n!}\,a_0,$$

which implies that the solution of the differential equation is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{c^n}{n!}\,x^n \quad \Rightarrow \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c\,x)^n}{n!} \quad \Rightarrow \quad y(x) = a_0\,e^{-c\,x}.$$

◁

Example 3.1.3. Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

Solution: We know that the solution can be found computing the roots of the characteristic polynomial $r^2 + 1 = 0$, which gives us the solutions

$$y(x) = a_0\,\cos(x) + a_1\,\sin(x).$$

We now recover this solution using the power series,

$$y = \sum_{n=0}^{\infty} a_n\,x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n\,x^{(n-1)}, \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n\,x^{(n-2)}.$$

Introduce the expressions above into the differential equation, which involves only the function and its second derivative,

$$\sum_{n=2}^{\infty} n(n-1) a_n\,x^{n-2} + \sum_{n=0}^{\infty} a_n\,x^n = 0.$$

Relabel the first sum above, so that both sums have the same factor x^n. One way is,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)}\,x^n + \sum_{n=0}^{\infty} a_n\,x^n = 0.$$

Now we can write both sums using one single sum as follows,

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1) a_{(n+2)} + a_n\big]\,x^n = 0 \quad \Rightarrow \quad (n+2)(n+1) a_{(n+2)} + a_n = 0. \quad n \geqslant 0.$$

The last equation is the *recurrence relation*. The solution of this relation can again be found by writing down the first few cases, and we start with even values of n, that is,

$$\begin{aligned}
n &= 0, & (2)(1)a_2 &= -a_0 & &\Rightarrow & a_2 &= -\frac{1}{2!}\,a_0,\\
n &= 2, & (4)(3)a_4 &= -a_2 & &\Rightarrow & a_4 &= \frac{1}{4!}\,a_0,\\
n &= 4, & (6)(5)a_6 &= -a_4 & &\Rightarrow & a_6 &= -\frac{1}{6!}\,a_0.
\end{aligned}$$

One can check that the even coefficients a_{2k} can be written as

$$a_{2k} = \frac{(-1)^k}{(2k)!}\,a_0.$$

The coefficients a_n for the odd values of n can be found in the same way, that is,

$$\begin{aligned}
n &= 1, & (3)(2)a_3 &= -a_1 & &\Rightarrow & a_3 &= -\frac{1}{3!}\,a_1,\\
n &= 3, & (5)(4)a_5 &= -a_3 & &\Rightarrow & a_5 &= \frac{1}{5!}\,a_1,\\
n &= 5, & (7)(6)a_7 &= -a_5 & &\Rightarrow & a_7 &= -\frac{1}{7!}\,a_1.
\end{aligned}$$

One can check that the odd coefficients a_{2k+1} can be written as

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\,a_1.$$

Split the sum in the expression for y into even and odd sums. We have the expression for the even and odd coefficients. Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\,x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\,x^{2k+1}.$$

One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0\,\cos(x) + a_1\,\sin(x).$$

$\lhd$

Example 3.1.4. Find the first four terms of the power series expansion around the point $x_0 = 1$ of each fundamental solution to the differential equation

$$y'' - x\,y' - y = 0.$$

Solution: This is a differential equation we cannot solve with the methods of previous sections. This is a second order, variable coefficients equation. We use the power series method, so we look for solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

We start working in the middle term in the differential equation. Since the power series is centered at $x_0 = 1$, it is convenient to re-write this term as $x\,y' = [(x-1)+1]\,y'$, that is,

$$\begin{aligned} x\,y' &= \sum_{n=1}^{\infty} n a_n x (x-1)^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n \big[(x-1)+1\big](x-1)^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}. \end{aligned} \tag{3.1.4}$$

As usual by now, the first sum on the right-hand side of Eq. (3.1.4) can start at $n = 0$, since we are only adding a zero term to the sum, that is,

$$\sum_{n=1}^{\infty} n a_n (x-1)^n = \sum_{n=0}^{\infty} n a_n (x-1)^n;$$

while it is convenient to relabel the second sum in Eq. (3.1.4) follows,

$$\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{(n+1)} (x-1)^n;$$

so both sums in Eq. (3.1.4) have the same factors $(x-1)^n$. We obtain the expression

$$\begin{aligned} x\,y' &= \sum_{n=0}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{(n+1)} (x-1)^n \\ &= \sum_{n=0}^{\infty} \big[n a_n + (n+1) a_{(n+1)}\big](x-1)^n. \end{aligned} \tag{3.1.5}$$

In a similar way relabel the index in the expression for y'', so we obtain

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-1)^n. \tag{3.1.6}$$

If we use Eqs. (3.1.5)-(3.1.6) in the differential equation, together with the expression for y, the differential equation can be written as follows

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-1)^n - \sum_{n=0}^{\infty} \big[n a_n + (n+1) a_{(n+1)}\big](x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0.$$

We can now put all the terms above into a single sum,

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1) a_{(n+2)} - (n+1) a_{(n+1)} - n a_n - a_n\Big]\,(x-1)^n = 0.$$

This expression provides the *recurrence relation* for the coefficients a_n with $n \geqslant 0$, that is,

$$\begin{aligned} (n+2)(n+1) a_{(n+2)} - (n+1) a_{(n+1)} - (n+1) a_n &= 0 \\ (n+1)\Big[(n+2) a_{(n+2)} - a_{(n+1)} - a_n\Big] &= 0, \end{aligned}$$

which can be rewritten as follows,

$$(n+2) a_{(n+2)} - a_{(n+1)} - a_n = 0. \tag{3.1.7}$$

We can solve this recurrence relation for the first four coefficients,

$$\begin{aligned}
&n=0 && 2a_2-a_1-a_0=0 && \Rightarrow && a_2=\frac{a_1}{2}+\frac{a_0}{2},\\
&n=1 && 3a_3-a_2-a_1=0 && \Rightarrow && a_3=\frac{a_1}{2}+\frac{a_0}{6},\\
&n=2 && 4a_4-a_3-a_2=0 && \Rightarrow && a_4=\frac{a_1}{4}+\frac{a_0}{6}.
\end{aligned}$$

Therefore, the first terms in the power series expression for the solution y of the differential equation are given by

$$y=a_0+a_1(x-1)+\Big(\frac{a_0}{2}+\frac{a_1}{2}\Big)(x-1)^2+\Big(\frac{a_0}{6}+\frac{a_1}{2}\Big)(x-1)^3+\Big(\frac{a_0}{6}+\frac{a_1}{4}\Big)(x-1)^4+\cdots$$

which can be rewritten as

$$\begin{aligned}
y=\;&a_0\Big[1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots\Big]\\
&+a_1\Big[(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4+\cdots\Big]
\end{aligned}$$

So the first four terms on each fundamental solution are given by

$$\begin{aligned}
y_1&=1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4,\\
y_2&=(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4.
\end{aligned}$$

◁

Example 3.1.5. Find the first three terms of the power series expansion around the point $x_0=2$ of each fundamental solution to the differential equation

$$y''-x\,y=0.$$

Solution: We then look for solutions of the form

$$y=\sum_{n=0}^{\infty}a_n(x-2)^n.$$

It is convenient to rewrite the function $x\,y=[(x-2)+2]\,y$, that is,

$$\begin{aligned}
xy&=\sum_{n=0}^{\infty}a_nx(x-2)^n\\
&=\sum_{n=0}^{\infty}a_n\big[(x-2)+2\big](x-2)^n\\
&=\sum_{n=0}^{\infty}a_n(x-2)^{n+1}+\sum_{n=0}^{\infty}2a_n(x-2)^n. \qquad (3.1.8)
\end{aligned}$$

We now relabel the first sum on the right-hand side of Eq. (3.1.8) in the following way,

$$\sum_{n=0}^{\infty}a_n(x-2)^{n+1}=\sum_{n=1}^{\infty}a_{(n-1)}(x-2)^n. \qquad (3.1.9)$$

We do the same type of relabeling on the expression for y'',

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}(x-2)^n. \end{aligned}$$

Then, the differential equation above can be written as follows

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{(n+2)}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{(n-1)}(x-2)^n = 0$$

$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)}\Big]\,(x-2)^n = 0.$$

So the *recurrence relation* for the coefficients a_n is given by

$$a_2 - a_0 = 0, \qquad (n+2)(n+1)a_{(n+2)} - 2a_n - a_{(n-1)} = 0, \qquad n \geqslant 1.$$

We can solve this recurrence relation for the first four coefficients,

$$\begin{aligned} &n=0 & a_2 - a_0 &= 0 & &\Rightarrow & a_2 &= a_0, \\ &n=1 & (3)(2)a_3 - 2a_1 - a_0 &= 0 & &\Rightarrow & a_3 &= \frac{a_0}{6} + \frac{a_1}{3}, \\ &n=2 & (4)(3)a_4 - 2a_2 - a_1 &= 0 & &\Rightarrow & a_4 &= \frac{a_0}{6} + \frac{a_1}{12}. \end{aligned}$$

Therefore, the first terms in the power series expression for the solution y of the differential equation are given by

$$y = a_0 + a_1(x-2) + a_0(x-2)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{3}\Big)(x-2)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{12}\Big)(x-2)^4 + \cdots$$

which can be rewritten as

$$\begin{aligned} y = {} & a_0\Big[1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \cdots\Big] \\ & + a_1\Big[(x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \cdots\Big] \end{aligned}$$

So the first three terms on each fundamental solution are given by

$$\begin{aligned} y_1 &= 1 + (x-2)^2 + \frac{1}{6}(x-2)^3, \\ y_2 &= (x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4. \end{aligned}$$

◁

3.1.3. The Legendre Equation. The Legendre equation appears when one solves the Laplace equation in spherical coordinates. The Laplace equation describes several phenomena, such as the static electric potential near a charged body, or the gravitational potential of a planet or star. When the Laplace equation describes a situation having spherical symmetry it makes sense to use spherical coordinates to solve the equation. It is in that case that the Legendre equation appears for a variable related to the polar angle in the spherical coordinate system. See Jackson's classic book on electrodynamics [8], § 3.1, for a derivation of the Legendre equation from the Laplace equation.

Example 3.1.6. Find all solutions of the Legendre equation

$$(1-x^2)\,y'' - 2x\,y' + l(l+1)\,y = 0,$$

where l is any real constant, using power series centered at $x_0 = 0$.

Solution: We start writing the equation in the form of Theorem 3.1.2,

$$y'' - \frac{2x}{(1-x^2)}\,y' + \frac{l(l+1)}{(1-x^2)}\,y = 0.$$

It is clear that the coefficient functions

$$p(x) = -\frac{2x}{(1-x^2)}, \qquad q(x) = \frac{l(l+1)}{(1-x^2)},$$

are analytic on the interval $|x| < 1$, which is centered at $x_0 = 0$. Theorem 3.1.2 says that there are two solutions linearly independent and analytic on that interval. So we write the solution as a power series centered at $x_0 = 0$,

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^n,$$

and we compute its derivative,

$$y'(x) = \sum_{n=0}^{\infty} na_n\,x^{n-1} = \sum_{n=1}^{\infty} na_n\,x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{(n+1)}\,x^n,$$

where the first equality is the plain derivative, in the second we start the sum at $n = 1$ since the first term in the sum is zero, and in the third equality we rename the summation index $n \to n-1$, so when the old index starts at one, the new index starts at zero. The second derivative of y is treated in a similar way,

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n\,x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}\,x^n.$$

Then we continue working as follows,

$$\begin{aligned}
y'' &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}\,x^n,\\
-x^2\,y'' &= \sum_{n=0}^{\infty} -(n-1)na_n\,x^n,\\
-2x\,y' &= \sum_{n=0}^{\infty} -2na_n\,x^n,\\
l(l+1)\,y &= \sum_{n=0}^{\infty} l(l+1)a_n\,x^n.
\end{aligned}$$

The Legendre equation says that the addition of the four equations above must be zero,

$$\sum_{n=0}^{\infty} \big((n+2)(n+1)a_{(n+2)} - (n-1)na_n - 2na_n + l(l+1)a_n\big)\,x^n = 0.$$

Therefore, every term in that sum must vanish,

$$(n+2)(n+1)a_{(n+2)} - (n-1)na_n - 2na_n + l(l+1)a_n = 0, \qquad n \geqslant 0.$$

This is the recurrence relation for the coefficients a_n. After a few manipulations the recurrence relation becomes

$$a_{(n+2)} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}\, a_n, \qquad n \geqslant 0.$$

By giving values to n we obtain,

$$a_2 = -\frac{l(l+1)}{2!}\, a_0, \quad a_3 = -\frac{(l-1)(l+2)}{3!}\, a_1.$$

Since a_4 is related to a_2 and a_5 is related to a_3, we get,

$$a_4 = -\frac{(l-2)(l+3)}{(3)(4)}\, a_2 \quad \Rightarrow \quad a_4 = \frac{(l-2)l(l+1)(l+3)}{4!}\, a_0,$$

$$a_5 = -\frac{(l-3)(l+4)}{(4)(5)}\, a_3 \quad \Rightarrow \quad a_5 = \frac{(l-3)(l-1)(l+2)(l+4)}{5!}\, a_1.$$

If one keeps solving the coefficients a_n in terms of either a_0 or a_1, one gets the expression,

$$\begin{aligned} y(x) = \quad & a_0 \Big[1 - \frac{l(l+1)}{2!}\, x^2 + \frac{(l-2)l(l+1)(l+3)}{4!}\, x^4 + \cdots \Big] \\ & + a_1 \Big[x - \frac{(l-1)(l+2)}{3!}\, x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!}\, x^5 + \cdots \Big]. \end{aligned}$$

Hence, the fundamental solutions are

$$\begin{aligned} y_1(x) &= 1 - \frac{l(l+1)}{2!}\, x^2 + \frac{(l-2)l(l+1)(l+3)}{4!}\, x^4 + \cdots \\ y_2(x) &= x - \frac{(l-1)(l+2)}{3!}\, x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!}\, x^5 + \cdots . \end{aligned}$$

The ration test provides the interval where the seires above converge. For function y_1 we get, replacing n by $2n$,

$$\Big|\frac{a_{2n+2}\, x^{2n+2}}{a_{2n}\, x^{2n}}\Big| = \Big|= \frac{(l-2n)(l+2n+1)}{(2n+1)(2n+2)}\Big|\, |x^2| \to |x|^2 \quad \text{as} \quad n \to \infty.$$

A similar result holds for y_2. So both series converge on the interval defined by $|x| < 1$. $\lhd$

Remark: The functions y_1, y_2 are called Legendre functions. For a non-integer value of the constant l these functions cannot be written in terms of elementary functions. But when l is an integer, one of these series terminate and becomes a polynomial. The case l being a nonnegative integer is specially relevant in physics. For l even the function y_1 becomes a polynomial while y_2 remains an infinite series. For l odd the function y_2 becomes a polynomial while the y_1 remains an infinite series. For example, for $l = 0, 1, 2, 3$ we get,

$$\begin{aligned} l &= 0, & y_1(x) &= 1, \\ l &= 1, & y_2(x) &= x, \\ l &= 2, & y_1(x) &= 1 - 3x^2, \\ l &= 3, & y_2(x) &= x - \frac{5}{3}\, x^3. \end{aligned}$$

The Legendre polynomials are proportional to these polynomials. The proportionality factor for each polynomial is chosen so that the Legendre polynomials have unit length in a particular chosen inner product. We just say here that the first four polynomials are

$$\begin{array}{llll} l=0, & y_1(x)=1, & P_0=y_1, & P_0(x)=1,\\ l=1, & y_2(x)=x, & P_1=y_2, & P_1(x)=x,\\ l=2, & y_1(x)=1-3x^2, & P_2=-\dfrac{1}{2}\,y_1, & P_2(x)=\dfrac{1}{2}\big(3x^2-1\big),\\ l=3, & y_2(x)=x-\dfrac{5}{3}\,x^3, & P_3=-\dfrac{3}{2}\,y_2, & P_3(x)=\dfrac{1}{2}\big(5x^3-3x\big). \end{array}$$

These polynomials, P_n, are called Legendre polynomials. The graph of the first four Legendre polynomials is given in Fig. 1.

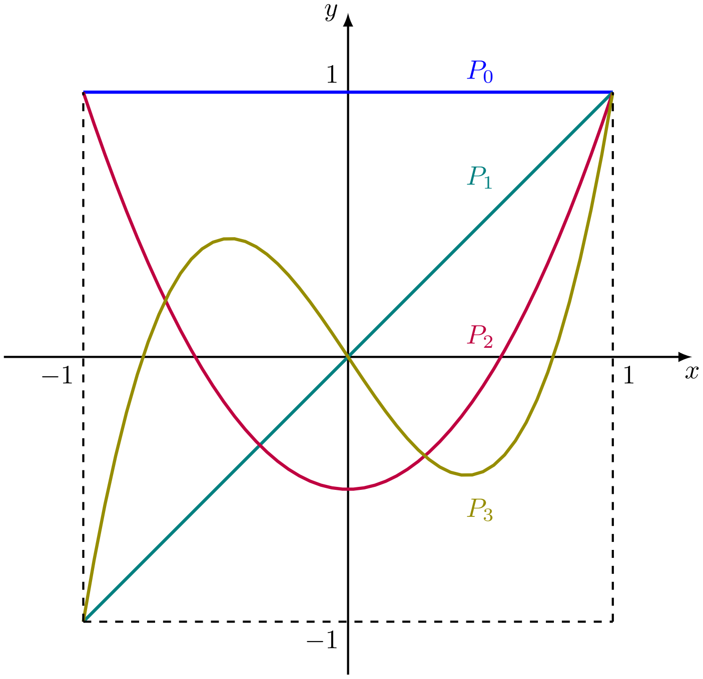


Figure 1. The graph of the first four Legendre polynomials.

3.1.4. Exercises.

3.1.1.- .

3.1.2.- .

3.2. Solutions Near Regular Singular Points

We continue with our study of the solutions to the differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0.$$

In § 3.1 we studied the case where the coefficient functions p and q were analytic functions. We saw that the solutions were also analytic and we used power series to find them. In § **??** we studied the case where the coefficients p and q were singular at a point x_0. The singularity was of a very particular form,

$$p(x) = \frac{p_0}{(x - x_0)}, \qquad q(x) = \frac{q_0}{(x - x_0)^2},$$

where p_0, q_0 are constants. The equation was called the Euler equidimensional equation. We found solutions near the singular point x_0. We found out that some solutions were analytic at x_0 and some solutions were singular at x_0. In this section we study equations with coefficients p and q being again singular at a point x_0. The singularity in this case is such that both functions below

$$(x - x_0)p(x), \qquad (x - x_0)^2 q(x)$$

are analytic in a neighborhood of x_0. The Euler equation is the particular case where these functions above are constants. Now we say they admit power series expansions centered at x_0. So we study equations that are close to Euler equations when the variable x is close to the singular point x_0. We will call the point x_0 a regular singular point. That is, a singular point that is not so singular. We will find out that some solutions may be well defined at the regular singular point and some other solutions may be singular at that point.

3.2.1. Regular Singular Points.

In § 3.1 we studied second order equations

$$y'' + p(x)\, y' + q(x)\, y = 0.$$

and we looked for solutions near regular points of the equation. A point x_0 is a regular point of the equation iff the functions p and q are analytic in a neighborhood of x_0. In particular the definition means that these functions have power series centered at x_0,

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

which converge in a neighborhood of x_0. A point x_0 is called a singular point of the equation if the coefficients p and q are not analytic on any set containing x_0. In this section we study a particular type of singular points. We study singular points that are not so singular.

Definition 3.2.1. *A point $x_0 \in \mathbb{R}$ is a* ***regular singular point*** *of the equation*

$$y'' + p(x)\, y' + q(x)\, y = 0.$$

iff both functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ are analytic on a neighborhood containing x_0, where

$$\tilde{p}_{x_0}(x) = (x - x_0)p(x), \qquad \tilde{q}_{x_0}(x) = (x - x_0)^2 q(x).$$

Remark: The singular point x_0 in an Euler equidimensional equation is regular singular. In fact, the functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ are not only analytic, they are actually constant. The proof is simple, take the Euler equidimensional equation

$$y'' + \frac{p_0}{(x - x_0)}\, y' + \frac{q_0}{(x - x_0)^2}\, y = 0,$$

and compute the functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ for the point x_0,

$$\tilde{p}_{x_0}(x) = (x - x_0)\Big(\frac{p_0}{(x - x_0)}\Big) = p_0, \qquad \tilde{q}_{x_0}(x) = (x - x_0)^2\Big(\frac{q_0}{(x - x_0)^2}\Big) = q_0.$$

Example 3.2.1. Show that the singular point of Euler equation below is regular singular,

$$(x-3)^2\, y'' + 2(x-3)\, y' + 4\, y = 0.$$

Solution: Divide the equation by $(x-3)^2$, so we get the equation in the standard form

$$y'' + \frac{2}{(x-3)}\, y' + \frac{4}{(x-3)^2}\, y = 0.$$

The functions p and q are given by

$$p(x) = \frac{2}{(x-3)}, \qquad q(x) = \frac{4}{(x-3)^2}.$$

The functions $\tilde{p}_3$ and $\tilde{q}_3$ for the point $x_0 = 3$ are constants,

$$\tilde{p}_3(x) = (x-3)\Big(\frac{2}{(x-3)}\Big) = 2, \qquad \tilde{q}_3(x) = (x-3)^2\Big(\frac{4}{(x-3)^2}\Big) = 4.$$

Therefore they are analytic. This shows that $x_0 = 3$ is regular singular. ◁

Example 3.2.2. Find the regular-singular points of the Legendre equation

$$(1-x^2)\, y'' - 2x\, y' + l(l+1)\, y = 0,$$

where l is a real constant.

Solution: We start writing the Legendre equation in the standard form

$$y'' - \frac{2x}{(1-x^2)}\, y' + \frac{l(l+1)}{(1-x^2)}\, y = 0,$$

The functions p and q are given by

$$p(x) = -\frac{2x}{(1-x^2)}, \qquad q(x) = \frac{l(l+1)}{(1-x^2)}.$$

These functions are analytic except where the denominators vanish.

$$(1-x^2) = (1-x)(1+x) = 0 \quad \Rightarrow \quad \begin{cases} x_0 = 1, \\ x_1 = -1. \end{cases}$$

Let us start with the singular point $x_0 = 1$. The functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ for this point are,

$$\tilde{p}_{x_0}(x) = (x-1)p(x) = (x-1)\Big(-\frac{2x}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{p}_{x_0}(x) = \frac{2x}{(1+x)},$$

$$\tilde{q}_{x_0}(x) = (x-1)^2 q(x) = (x-1)^2\Big(\frac{l(l+1)}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_0}(x) = -\frac{l(l+1)(x-1)}{(1+x)}.$$

These two functions are analytic in a neighborhood of $x_0 = 1$. (Both $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ have no vertical asymptote at $x_0 = 1$.) Therefore, the point $x_0 = 1$ is a regular singular point. We now need to do a similar calculation with the point $x_1 = -1$. The functions $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ for this point are,

$$\tilde{p}_{x_1}(x) = (x+1)p(x) = (x+1)\Big(-\frac{2x}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{p}_{x_1}(x) = -\frac{2x}{(1-x)},$$

$$\tilde{q}_{x_1}(x) = (x+1)^2 q(x) = (x+1)^2\Big(\frac{l(l+1)}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_1}(x) = \frac{l(l+1)(x+1)}{(1-x)}.$$

These two functions are analytic in a neighborhood of $x_1 = -1$. (Both $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ have no vertical asymptote at $x_1 = -1$.) Therefore, the point $x_1 = -1$ is a regular singular point.◁

Example 3.2.3. Find the regular singular points of the differential equation

$$(x+2)^2(x-1)\, y'' + 3(x-1)\, y' + 2\, y = 0.$$

Solution: We start writing the equation in the standard form

$$y'' + \frac{3}{(x+2)^2}\, y' + \frac{2}{(x+2)^2(x-1)}\, y = 0.$$

The functions p and q are given by

$$p(x) = \frac{3}{(x+2)^2}, \qquad q(x) = \frac{2}{(x+2)^2(x-1)}.$$

The denominators of the functions above vanish at $x_0 = -2$ and $x_1 = 1$. These are singular points of the equation. Let us find out whether these singular points are regular singular or not. Let us start with $x_0 = -2$. The functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ for this point are,

$$\tilde{p}_{x_0}(x) = (x+2)p(x) = (x+2)\Big(\frac{3}{(x+2)^2}\Big) \quad \Rightarrow \quad \tilde{p}_{x_0}(x) = \frac{3}{(x+2)},$$

$$\tilde{q}_{x_0}(x) = (x+2)^2 q(x) = (x+2)^2\Big(\frac{2}{(x+2)^2(x-1)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_0}(x) = -\frac{2}{(x-1)}.$$

We see that $\tilde{q}_{x_0}$ is analytic on a neighborhood of $x_0 = -2$, but $\tilde{p}_{x_0}$ is not analytic on any neighborhood containing $x_0 = -2$, because the function $\tilde{p}_{x_0}$ has a vertical asymptote at $x_0 = -2$. So the point $x_0 = -2$ is not a regular singular point. We need to do a similar calculation for the singular point $x_1 = 1$. The functions $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ for this point are,

$$\tilde{p}_{x_1}(x) = (x-1)p(x) = (x-1)\Big(\frac{3}{(x+2)^2}\Big) \quad \Rightarrow \quad \tilde{p}_{x_1}(x) = \frac{3(x-1)}{(x+2)},$$

$$\tilde{q}_{x_1}(x) = (x-1)^2 q(x) = (x-1)^2\Big(\frac{2}{(x+2)^2(x-1)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_1}(x) = -\frac{2(x-1)}{(x+2)}.$$

We see that both functions $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ are analytic on a neighborhood containing $x_1 = 1$. (Both $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ have no vertical asymptote at $x_1 = 1$.) Therefore, the point $x_1 = 1$ is a regular singular point. ◁

Remark: It is fairly simple to find the regular singular points of an equation. Take the equation in out last example, written in standard form,

$$y'' + \frac{3}{(x+2)^2}\, y' + \frac{2}{(x+2)^2(x-1)}\, y = 0.$$

The functions p and q are given by

$$p(x) = \frac{3}{(x+2)^2}, \qquad q(x) = \frac{2}{(x+2)^2(x-1)}.$$

The singular points are given by the zeros in the denominators, that is $x_0 = -2$ and $x_1 = 1$. The point x_0 is not regular singular because function p diverges at $x_0 = -2$ faster than $\dfrac{1}{(x+2)}$. The point $x_1 = 1$ is regular singular because function p is regular at $x_1 = 1$ and function q diverges at $x_1 = 1$ slower than $\dfrac{1}{(x-1)^2}$.

3.2.2. The Frobenius Method.

We now assume that the differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0, \tag{3.2.1}$$

has a regular singular point. We want to find solutions to this equation that are defined arbitrary close to that regular singular point. Recall that a point x_0 is a regular singular point of the equation above iff the functions $(x - x_0)\,p$ and $(x - x_0)^2\,q$ are analytic at x_0. A function is analytic at a point iff it has a convergent power series expansion in a neighborhood of that point. In our case this means that near a regular singular point holds

$$(x - x_0)\,p(x) = \sum_{n=0}^{\infty} p_n\,(x - x_0)^n = p_0 + p_1(x - x_0) + p_2(x - x_0)^2 + \cdots$$

$$(x - x_0)^2\,q(x) = \sum_{n=0}^{\infty} q_n\,(x - x_0)^n = q_0 + q_1(x - x_0) + q_2(x - x_0)^2 + \cdots$$

This means that near x_0 the function p diverges at most like $(x - x_0)^{-1}$ and function q diverges at most like $(x - x_0)^{-2}$, as it can be seen from the equations

$$p(x) = \frac{p_0}{(x - x_0)} + p_1 + p_2(x - x_0) + \cdots$$

$$q(x) = \frac{q_0}{(x - x_0)^2} + \frac{q_1}{(x - x_0)} + q_2 + \cdots$$

Therefore, for p_0 and q_0 nonzero and x close to x_0 we have the relations

$$p(x) \simeq \frac{p_0}{(x - x_0)}, \qquad q(x) \simeq \frac{q_0}{(x - x_0)^2}, \qquad x \simeq x_0,$$

where the symbol $a \simeq b$, with $a,\, b \in \mathbb{R}$ means that $|a - b|$ is close to zero. In other words, the for x close to a regular singular point x_0 the coefficients of Eq. (3.2.1) are close to the coefficients of the Euler equidimensional equation

$$(x - x_0)^2\,y_e'' + p_0(x - x_0)\,y_e' + q_0\,y_e = 0,$$

where p_0 and q_0 are the zero order terms in the power series expansions of $(x - x_0)\,p$ and $(x - x_0)^2\,q$ given above. One could expect that solutions y to Eq. (3.2.1) are close to solutions y_e to this Euler equation. One way to put this relation in a more precise way is

$$y(x) = y_e(x) \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \Rightarrow \quad y(x) = y_e(x)\,\big(a_0 + a_1(x - x_0) + \cdots\big).$$

Recalling that at least one solution to the Euler equation has the form $y_e(x) = (x - x_0)^r$, where r is a root of the indicial polynomial

$$r(r - 1) + p_0 r + q_0 = 0,$$

we then expect that for x close to x_0 the solution to Eq. (3.2.1) be close to

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

This expression for the solution is usually written in a more compact way as follows,

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{(r+n)}.$$

This is the main idea of the Frobenius method to find solutions to equations with regular singular points. To look for solutions that are close to solutions to an appopriate Euler equation. We now state two theorems summarize a few formulas for solutions to differential equations with regular singular points.

Theorem 3.2.2 (Frobenius). *Assume that the differential equation*

$$y'' + p(x)\, y' + q(x)\, y = 0, \tag{3.2.2}$$

has a regular singular point $x_0 \in \mathbb{R}$ *and denote by* p_0, q_0 *the zero order terms in*

$$(x - x_0)\, p(x) = \sum_{n=0}^{\infty} p_n\, (x - x_0)^n, \qquad (x - x_0)^2\, q(x) = \sum_{n=0}^{\infty} q_n\, (x - x_0)^n.$$

Let r_+, r_- *be the solutions of the indicial equation*

$$r(r-1) + p_0\, r + q_0 = 0.$$

(a) If $(r_+ - r_-)$ *is not an integer, then the differential equation in (3.2.2) has two independent solutions* y_+, y_- *of the form*

$$y_+(x) = |x - x_0|^{r_+} \sum_{n=0}^{\infty} a_n\, (x - x_0)^n, \quad \text{with} \quad a_0 = 1,$$

$$y_-(x) = |x - x_0|^{r_-} \sum_{n=0}^{\infty} b_n\, (x - x_0)^n, \quad \text{with} \quad b_0 = 1.$$

(b) If $(r_+ - r_-) = N$*, a nonnegative integer, then the differential equation in (3.2.2) has two independent solutions* y_+, y_- *of the form*

$$y_+(x) = |x - x_0|^{r_+} \sum_{n=0}^{\infty} a_n\, (x - x_0)^n, \quad \text{with} \quad a_0 = 1,$$

$$y_-(x) = |x - x_0|^{r_-} \sum_{n=0}^{\infty} b_n\, (x - x_0)^n + c\, y_+(x)\, \ln|x - x_0|, \quad \text{with} \quad b_0 = 1.$$

The constant c *is nonzero if* $N = 0$*. If* $N > 0$*, the constant* c *may or may not be zero.*

In both cases above the series converge in the interval defined by $|x - x_0| < \rho$ *and the differential equation is satisfied for* $0 < |x - x_0| < \rho$*.*

Remarks:

(a) The statements above are taken from Apostol's second volume [2], Theorems 6.14, 6.15. For a sketch of the proof see Simmons [10]. A proof can be found in [5, 7].

(b) The existence of solutions and their behavior in a neighborhood of a singular point was first shown by Lazarus Fuchs in 1866. The construction of the solution using singular power series expansions was first shown by Ferdinand Frobenius in 1874.

We now give a summary of the Frobenius method to find the solutions mentioned in Theorem 3.2.2 to a differential equation having a regular singular point. For simplicity we only show how to obtain the solution y_+.

(1) Look for a solution y of the form $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n\, (x - x_0)^{(n+r)}$.

(2) Introduce this power series expansion into the differential equation and find the indicial equation for the *exponent* r. Find the larger solution of the indicial equation.

(3) Find a *recurrence relation* for the coefficients a_n.

(4) Introduce the larger root r into the recurrence relation for the coefficients a_n. Only then, solve this latter recurrence relation for the coefficients a_n.

(5) Using this procedure we will find the solution y_+ in Theorem 3.2.2.

We now show how to use these steps to find one solution of a differential equation near a regular singular point. We show the case where the roots of the indicial polynomial differ by an integer. We show that in this case we obtain only solution y_+. The solution y_- does not have the form $y(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^{(n+r)}$. Theorem 3.2.2 says that there is a logarithmic term in the solution. We do not compute that solution here.

Example 3.2.4. Find the solution y near the regular singular point $x_0 = 0$ of the equation

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: We look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^{(n+r)}.$$

The first and second derivatives are given by

$$y'(x) = \sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r-1)}, \qquad y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r-2)}.$$

In the case $r = 0$ we had the relation $\sum_{n=0}^{\infty} na_n\,x^{(n-1)} = \sum_{n=1}^{\infty} na_n\,x^{(n-1)}$. But in our case $r \neq 0$, so we do not have the freedom to change in this way the starting value of the summation index n. If we want to change the initial value for n, we have to re-label the summation index. We now introduce these expressions into the differential equation. It is convenient to do this step by step. We start with the term $(x+3)y$, which has the form,

$$\begin{aligned}(x+3)\,y &= (x+3)\sum_{n=0}^{\infty} a_n\,x^{(n+r)}\\ &= \sum_{n=0}^{\infty} a_n\,x^{(n+r+1)} + \sum_{n=0}^{\infty} 3a_n\,x^{(n+r)}\\ &= \sum_{n=1}^{\infty} a_{(n-1)}\,x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n\,x^{(n+r)}.\end{aligned} \tag{3.2.3}$$

We continue with the term containing y',

$$\begin{aligned}-x(x+3)\,y' &= -(x^2+3x)\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r-1)}\\ &= -\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r+1)} - \sum_{n=0}^{\infty}3(n+r)a_n\,x^{(n+r)}\\ &= -\sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\,x^{(n+r)} - \sum_{n=0}^{\infty}3(n+r)a_n\,x^{(n+r)}.\end{aligned} \tag{3.2.4}$$

Then, we compute the term containing y'' as follows,

$$\begin{aligned}x^2\,y'' &= x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r-2)}\\ &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r)}.\end{aligned} \tag{3.2.5}$$

As one can see from Eqs.(3.2.3)-(3.2.5), the guiding principle to rewrite each term is to have the power function $x^{(n+r)}$ labeled in the same way on every term. For example, in Eqs.(3.2.3)-(3.2.5) we do not have a sum involving terms with factors $x^{(n+r-1)}$ or factors $x^{(n+r+1)}$. Then, the differential equation can be written as follows,

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r)} - \sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\,x^{(n+r)}$$
$$-\sum_{n=0}^{\infty}3(n+r)a_n\,x^{(n+r)} + \sum_{n=1}^{\infty}a_{(n-1)}\,x^{(n+r)} + \sum_{n=0}^{\infty}3a_n\,x^{(n+r)} = 0.$$

In the equation above we need to split the sums containing terms with $n \geqslant 0$ into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$, that is,

$$\big[r(r-1) - 3r + 3\big]a_0x^r +$$
$$\sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big]\,x^{(n+r)} = 0,$$

and this expression can be rewritten as follows,

$$\big[r(r-1) - 3r + 3\big]a_0x^r +$$
$$\sum_{n=1}^{\infty}\Big[\big[(n+r)(n+r-1) - 3(n+r) + 3\big]a_n - (n+r-1-1)a_{(n-1)}\Big]\,x^{(n+r)} = 0$$

and then,

$$\big[r(r-1) - 3r + 3\big]a_0x^r +$$
$$\sum_{n=1}^{\infty}\Big[\big[(n+r)(n+r-1) - 3(n+r-1)\big]a_n - (n+r-2)a_{(n-1)}\Big]\,x^{(n+r)} = 0$$

hence,

$$\big[r(r-1) - 3r + 3\big]a_0x^r + \sum_{n=1}^{\infty}\Big[(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)}\Big]\,x^{(n+r)} = 0.$$

The *indicial equation* and the *recurrence relation* are given by the equations

$$r(r-1) - 3r + 3 = 0, \tag{3.2.6}$$
$$(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)} = 0. \tag{3.2.7}$$

The way to solve these equations in (3.2.6)-(3.2.7) is the following: First, solve Eq. (3.2.6) for the exponent r, which in this case has two solutions $r_\pm$; second, introduce the first solution r_+ into the recurrence relation in Eq. (3.2.7) and solve for the coefficients a_n; the result is a solution y_+ of the original differential equation; then introduce the second solution r_- into Eq. (3.2.7) and solve again for the coefficients a_n; the new result is a second solution y_-. Let us follow this procedure in the case of the equations above:

$$r^2 - 4r + 3 = 0 \quad\Rightarrow\quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big] \quad\Rightarrow\quad \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Introducing the value $r_+ = 3$ into Eq. (3.2.7) we obtain

$$(n+2)n\,a_n - (n+1)a_{n-1} = 0.$$

One can check that the solution y_+ obtained form this recurrence relation is given by

$$y_+(x) = a_0\,x^3\Big[1 + \frac{2}{3}\,x + \frac{1}{4}\,x^2 + \frac{1}{15}\,x^3 + \cdots\Big].$$

Notice that $r_+ - r_- = 3 - 1 = 2$, this is a nonpositive integer. Theorem 3.2.2 says that the solution y_- contains a logarithmic term. Therefore, the solution y_- is not of the form $\sum_{n=0}^{\infty} a_n x^{(r+n)}$, as we have assumed in the calculations done in this example. But, what does happen if we continue this calculation for $r_- = 1$? What solution do we get? Let us find out. We introduce the value $r_- = 1$ into Eq. (3.2.7), then we get

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

One can also check that the solution $\tilde{y}_-$ obtained form this recurrence relation is given by

$$\begin{aligned} \tilde{y}_-(x) &= a_2\, x \Big[x^2 + \frac{2}{3}\, x^3 + \frac{1}{4}\, x^4 + \frac{1}{15}\, x^5 + \cdots\Big], \\ &= a_2\, x^3 \Big[1 + \frac{2}{3}\, x + \frac{1}{4}\, x^2 + \frac{1}{15}\, x^3 + \cdots\Big] \quad \Rightarrow \quad \tilde{y}_- = \frac{a_2}{a_1}\, y_+. \end{aligned}$$

So get a solution, but this solution is proportional to y_+. To get a solution not proportional to y_+ we need to add the logarithmic term, as in Theorem 3.2.2. ◁

3.2.3. The Bessel Equation. We saw in § 3.1 that the Legendre equation appears when one solves the Laplace equation in spherical coordinates. If one uses cylindrical coordinates insted, one needs to solve the Bessel equation. Recall we mentioned that the Laplace equation describes several phenomena, such as the static electric potential near a charged body, or the gravitational potential of a planet or star. When the Laplace equation describes a situation having cylindrical symmetry it makes sense to use cylindrical coordinates to solve it. Then the Bessel equation appears for the radial variable in the cylindrical coordinate system. See Jackson's classic book on electrodynamics [8], § 3.7, for a derivation of the Bessel equation from the Laplace equation.

The equation is named after Friedrich Bessel, a German astronomer from the first half of the seventeen century, who was the first person to calculate the distance to a star other than our Sun. Bessel's parallax of 1838 yielded a distance of 11 light years for the star 61 Cygni. In 1844 he discovered that Sirius, the brightest star in the sky, has a traveling companion. Nowadays such system is called a binary star. This companion has the size of a planet and the mass of a star, so it has a very high density, many thousand times the density of water. This was the first dead start discovered. Bessel first obtained the equation that now bears his name when he was studing star motions. But the equation first appeared in Daniel Bernoulli's studies of oscillations of a hanging chain. (Taken from Simmons' book [10], § 34.)

Example 3.2.5. Find all solutions $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, with $a_0 \neq 0$, of the Bessel equation

$$x^2\, y'' + x\, y' + (x^2 - \alpha^2)\, y = 0, \qquad x > 0,$$

where α is any real nonnegative constant, using the Frobenius method centered at $x_0 = 0$.

Solution: Let us double check that $x_0 = 0$ is a regular singular point of the equation. We start writing the equation in the standard form,

$$y'' + \frac{1}{x}\, y' + \frac{(x^2 - \alpha^2)}{x^2}\, y = 0,$$

so we get the functions $p(x) = 1/x$ and $q(x) = (x^2 - \alpha^2)/x^2$. It is clear that $x_0 = 0$ is a singular point of the equation. Since the functions

$$\tilde{p}(x) = xp(x) = 1, \qquad \tilde{q}(x) = x^2 q(x) = (x^2 - \alpha^2)$$

are analytic, we conclude that $x_0 = 0$ is a regular singular point. So it makes sense to look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}, \qquad x > 0.$$

We now compute the different terms needed to write the differential equation. We need,

$$x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} \quad \Rightarrow \quad y(x) = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)},$$

where we did the relabeling $n + 2 = m \to n$. The term with the first derivative is given by

$$x\, y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r)}.$$

The term with the second derivative has the form

$$x^2\, y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}.$$

Therefore, the differential equation takes the form

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} - \sum_{n=0}^{\infty} \alpha^2\, a_n x^{(n+r)} = 0.$$

Group together the sums that start at $n = 0$,

$$\sum_{n=0}^{\infty} \big[(n+r)(n+r-1) + (n+r) - \alpha^2\big] a_n\, x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)},$$

and cancel a few terms in the first sum,

$$\sum_{n=0}^{\infty} \big[(n+r)^2 - \alpha^2\big] a_n\, x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} = 0.$$

Split the sum that starts at $n = 0$ into its first two terms plus the rest,

$$(r^2 - \alpha^2) a_0\, x^r + \big[(r+1)^2 - \alpha^2\big] a_1\, x^{(r+1)} + \sum_{n=2}^{\infty} \big[(n+r)^2 - \alpha^2\big] a_n\, x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)}\, x^{(n+r)} = 0.$$

The reason for this splitting is that now we can write the two sums as one,

$$(r^2 - \alpha^2) a_0\, x^r + \big[(r+1)^2 - \alpha^2\big] a_1\, x^{(r+1)} + \sum_{n=2}^{\infty} \big\{ \big[(n+r)^2 - \alpha^2\big] a_n + a_{(n-2)} \big\}\, x^{(n+r)} = 0.$$

We then conclude that each term must vanish,

$$(r^2 - \alpha^2) a_0 = 0, \quad \big[(r+1)^2 - \alpha^2\big] a_1 = 0, \quad \big[(n+r)^2 - \alpha^2\big] a_n + a_{(n-2)} = 0, \quad n \geqslant 2. \tag{3.2.8}$$

This is the recurrence relation for the Bessel equation. It is here where we use that we look for solutions with $a_0 \neq 0$. In this example we do not look for solutions with $a_1 \neq 0$. Maybe it is a good exercise for the reader to find such solutions. But in this example we look for solutions with $a_0 \neq 0$. This condition and the first equation above imply that

$$r^2 - \alpha^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\alpha,$$

and recall that α is a nonnegative but otherwise arbitrary real number. The choice $r = r_+$ will lead to a solution y_α, and the choice $r = r_-$ will lead to a solution $y_{-\alpha}$. These solutions may or may not be linearly independent. This depends on the value of α, since $r_+ - r_- = 2\alpha$. One must be careful to study all possible cases.

Remark: Let us start with a very particular case. Suppose that both equations below hold,

$$(r^2 - \alpha^2) = 0, \qquad \big[(r+1)^2 - \alpha^2\big] = 0.$$

This equations are the result of both $a_0 \neq 0$ and $a_1 \neq 0$. These equations imply

$$r^2 = (r+1)^2 \quad \Rightarrow \quad 2r + 1 = 0 \quad \Rightarrow \quad r = -\frac{1}{2}.$$

But recall that $r = \pm\alpha$, and $\alpha \geqslant 0$, hence the case $a_0 \neq 0$ and $a_1 \neq 0$ happens only when $\alpha = 1/2$ and we choose $r_- = -\alpha = -1/2$. We leave computation of the solution $y_{-1/2}$ as an exercise for the reader. But the answer is

$$y_{-1/2}(x) = a_0\,\frac{\cos(x)}{\sqrt{x}} + a_1\,\frac{\sin(x)}{\sqrt{x}}.$$

From now on we assume that $\alpha \neq 1/2$. This condition on α, the equation $r^2 - \alpha^2 = 0$, and the remark above imply that

$$(r+1)^2 - \alpha^2 \neq 0.$$

So the second equation in the recurrence relation in (3.2.8) implies that $a_1 = 0$. Summarizing, the first two equations in the recurrence relation in (3.2.8) are satisfied because

$$r_\pm = \pm\alpha, \qquad a_1 = 0.$$

We only need to find the coefficients a_n, for $n \geqslant 2$ such that the third equation in the recurrence relation in (3.2.8) is satisfied. But we need to consider two cases, $r = r_+ = \alpha$ and $r_- = -\alpha$.

We start with the case $r = r_+ = \alpha$, and we get

$$(n^2 + 2n\alpha)\,a_n + a_{(n-2)} = 0 \quad \Rightarrow \quad n(n+2\alpha)\,a_n = -a_{(n-2)}.$$

Since $n \geqslant 2$ and $\alpha \geqslant 0$, the factor $(n + 2\alpha)$ never vanishes and we get

$$a_n = -\frac{a_{(n-2)}}{n(n+2\alpha)}.$$

This equation and $a_1 = 0$ imply that all coefficients $a_{2k+1} = 0$ for $k \geqslant 0$, the odd coefficients vanish. On the other hand, the even coefficent are nonzero. The coefficient a_2 is

$$a_2 = -\frac{a_0}{2(2+2\alpha)} \quad \Rightarrow \quad a_2 = -\frac{a_0}{2^2(1+\alpha)},$$

the coefficient a_4 is

$$a_4 = -\frac{a_2}{4(4+2\alpha)} = -\frac{a_2}{2^2(2)(2+\alpha)} \quad \Rightarrow \quad a_4 = \frac{a_0}{2^4(2)(1+\alpha)(2+\alpha)},$$

the coefficient a_6 is

$$a_6 = -\frac{a_4}{6(6+2\alpha)} = -\frac{a_4}{2^2(3)(3+\alpha)} \quad \Rightarrow \quad a_6 = -\frac{a_0}{2^6(3!)(1+\alpha)(2+\alpha)(3+\alpha)}.$$

Now it is not so hard to show that the general term a_{2k}, for $k = 0, 1, 2, \cdots$ has the form

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k}(k!)(1+\alpha)(2+\alpha)\cdots(k+\alpha)}.$$

We then get the solution y_α

$$y_\alpha(x) = a_0\, x^\alpha \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k!)(1+\alpha)(2+\alpha)\cdots(k+\alpha)}\Big], \qquad \alpha \geqslant 0. \tag{3.2.9}$$

The ratio test shows that this power series converges for all $x \geqslant 0$. When $a_0 = 1$ the corresponding solution is usually called in the literature as J_α,

$$J_\alpha(x) = x^\alpha \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k!)(1+\alpha)(2+\alpha)\cdots(k+\alpha)}\Big], \qquad \alpha \geqslant 0.$$

We now look for solutions to the Bessel equation coming from the choice $r = r_- = -\alpha$, with $a_1 = 0$, and $\alpha \neq 1/2$. The third equation in the recurrence relation in (3.2.8) implies

$$(n^2 - 2n\alpha)a_n + a_{(n-2)} = 0 \quad \Rightarrow \quad n(n-2\alpha)a_n = -a_{(n-2)}.$$

If $2\alpha = N$, a nonnegative integer, the second equation above implies that the recurrence relation cannot be solved for a_n with $n \geqslant N$. This case will be studied later on. Now assume that 2α is not a nonnegative integer. In this case the factor $(n - 2\alpha)$ never vanishes and

$$a_n = -\frac{a_{(n-2)}}{n(n-2\alpha)}.$$

This equation and $a_1 = 0$ imply that all coefficients $a_{2k+1} = 0$ for $k \geqslant 0$, the odd coefficients vanish. On the other hand, the even coefficent are nonzero. The coefficient a_2 is

$$a_2 = -\frac{a_0}{2(2-2\alpha)} \quad \Rightarrow \quad a_2 = -\frac{a_0}{2^2(1-\alpha)},$$

the coefficient a_4 is

$$a_4 = -\frac{a_2}{4(4-2\alpha)} = -\frac{a_2}{2^2(2)(2-\alpha)} \quad \Rightarrow \quad a_4 = \frac{a_0}{2^4(2)(1-\alpha)(2-\alpha)},$$

the coefficient a_6 is

$$a_6 = -\frac{a_4}{6(6-2\alpha)} = -\frac{a_4}{2^2(3)(3-\alpha)} \quad \Rightarrow \quad a_6 = -\frac{a_0}{2^6(3!)(1-\alpha)(2-\alpha)(3-\alpha)}.$$

Now it is not so hard to show that the general term a_{2k}, for $k = 0, 1, 2, \cdots$ has the form

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k}(k!)(1-\alpha)(2-\alpha)\cdots(k-\alpha)}.$$

We then get the solution $y_{-\alpha}$

$$y_{-\alpha}(x) = a_0\, x^\alpha \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k!)(1-\alpha)(2-\alpha)\cdots(k-\alpha)}\Big], \qquad \alpha \geqslant 0. \tag{3.2.10}$$

The ratio test shows that this power series converges for all $x \geqslant 0$. When $a_0 = 1$ the corresponding solution is usually called in the literature as $J_{-\alpha}$,

$$J_{-\alpha}(x) = x^{-\alpha} \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}(k!)(1-\alpha)(2-\alpha)\cdots(k-\alpha)}\Big], \qquad \alpha \geqslant 0.$$

The function $y_{-\alpha}$ was obtained assuming that 2α is not a nonnegative integer. From the calculations above it is clear that we need this condition on α so we can compute a_n in terms of $a_{(n-2)}$. Notice that $r_\pm = \pm\alpha$, hence $(r_+ - r_-) = 2\alpha$. So the condition on α is the condition $(r_+ - r_-)$ not a nonnegative integer, which appears in Theorem 3.2.2.

However, there is something special about the Bessel equation. That the constant 2α is not a nonnegative integer means that α is neither an integer nor an integer plus one-half. But the formula for $y_{-\alpha}$ is well defined even when α is an integer plus one-half, say

$k + 1/2$, for k integer. Introducing this $y_{-(k+1/2)}$ into the Bessel equation one can check that $y_{-(k+1/2)}$ is a solution to the Bessel equation.

Summarizing, the solutions of the Bessel equation function y_α is defined for every nonnegative real number α, and $y_{-\alpha}$ is defined for every nonnegative real number α except for nonnegative integers. For a given α such that both y_α and $y_{-\alpha}$ are defined, these functions are linearly independent. That these functions cannot be proportional to each other is simple to see, since for $\alpha > 0$ the function y_α is regular at the origin $x = 0$, while $y_{-\alpha}$ diverges.

The last case we need to study is how to find the solution $y_{-\alpha}$ when α is a nonnegative integer. We see that the expression in (3.2.10) is not defined when α is a nonnegative integer. And we just saw that this condition on α is a particular case of the condition in Theorem 3.2.2 that $(r_+ - r_-)$ is not a nonnegative integer. Theorem 3.2.2 gives us what is the expression for a second solution, $y_{-\alpha}$ linearly independent of y_α, in the case that α is a nonnegative integer. This expression is

$$y_{-\alpha}(x) = y_\alpha(x)\,\ln(x) + x^{-\alpha} \sum_{n=0}^{\infty} c_n x^n.$$

If we put this expression into the Bessel equation, one can find a recurrence relation for the coefficients c_n. This is a long calculation, and the final result is

$$\begin{aligned} y_{-\alpha}(x) &= y_\alpha(x)\,\ln(x) \\ &\quad - \frac{1}{2}\Big(\frac{x}{2}\Big)^{-\alpha} \sum_{n=0}^{\alpha-1} \frac{(\alpha-n-1)!}{n!}\Big(\frac{x}{2}\Big)^{2n} \\ &\quad - \frac{1}{2}\Big(\frac{x}{2}\Big)^{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{(h_n + h_{(n+\alpha)})}{n!\,(n+\alpha)!}\Big(\frac{x}{2}\Big)^{2n}, \end{aligned}$$

with $h_0 = 0$, $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ for $n \geqslant 1$, and α a nonnegative integer. $\triangleleft$

3.2.4. Exercises.

3.2.1.- .

3.2.2.- .

Notes on Chapter 3

Sometimes solutions to a differential equation cannot be written in terms of previously known functions. When that happens the we say that the solutions to the differential equation define a new type of functions. How can we work with, or let alone write down, a new function, a function that cannot be written in terms of the functions we already know? It is the differential equation what defines the function. So the function properties must be obtained from the differential equation itself. A way to compute the function values must come from the differential equation as well. The few paragraphs that follow try to give sense that this procedure is not as artificial as it may sound.

Differential Equations to Define Functions. We have seen in § 3.2 that the solutions of the Bessel equation for $\alpha \neq 1/2$ cannot be written in terms of simple functions, such as quotients of polynomials, trigonometric functions, logarithms and exponentials. We used power series including negative powers to write solutions to this equation. To study properties of these solutions one needs to use either the power series expansions or the equation itself. This type of study on the solutions of the Bessel equation is too complicated for these notes, but the interested reader can see [14].

We want to give an idea how this type of study can be carried out. We choose a differential equation that is simpler to study than the Bessel equation. We study two solutions, C and S, of this particular differential equation and we will show, using only the differential equation, that these solutions have all the properties that the cosine and sine functions have. So we will conclude that these solutions are in fact $C(x) = \cos(x)$ and $S(x) = \sin(x)$. This example is taken from Hassani's textbook [**?**], example 13.6.1, page 368.

Example 3.2.6. Let the function C be the unique solution of the initial value problem

$$C'' + C = 0, \qquad C(0) = 1, \quad C'(0) = 0,$$

and let the function S be the unique solution of the initial value problem

$$S'' + S = 0, \qquad S(0) = 0, \quad S'(0) = 1.$$

Use the differential equation to study these functions.

Solution:

(a) We start showing that these solutions C and S are linearly independent. We only need to compute their Wronskian at $x = 0$.

$$W(0) = C(0)\, S'(0) - C'(0)\, S(0) = 1 \neq 0.$$

Therefore the functions C and S are linearly independent.

(b) We now show that the function S is odd and the function C is even. The function $\hat{C}(x) = C(-x)$ satisfies the initial value problem

$$\hat{C}'' + \hat{C} = C'' + C = 0, \qquad \hat{C}(0) = C(0) = 1, \quad \hat{C}'(0) = -C'(0) = 0.$$

This is the same initial value problem satisfied by the function C. The uniqueness of solutions to these initial value problem implies that $C(-x) = C(x)$ for all $x \in \mathbb{R}$, hence the function C is even. The function $\hat{S}(x) = S(-x)$ satisfies the initial value problem

$$\hat{S}'' + \hat{S} = S'' + S = 0, \qquad \hat{S}(0) = S(0) = 0, \quad \hat{S}'(0) = -S'(0) = -1.$$

This is the same initial value problem satisfied by the function $-S$. The uniqueness of solutions to these initial value problem implies that $S(-x) = -S(x)$ for all $x \in \mathbb{R}$, hence the function S is odd.

(c) Next we find a differential relation between the functions C and S. Notice that the function $-C'$ is the unique solution of the initial value problem

$$(-C')'' + (-C') = 0, \qquad -C'(0) = 0, \quad (-C')'(0) = C(0) = 1.$$

This is precisely the same initial value problem satisfied by the function S. The uniqueness of solutions to these initial value problems implies that $-C = S$, that is for all $x \in \mathbb{R}$ holds

$$C'(x) = -S(x).$$

Take one more derivative in this relation and use the differential equation for C,

$$S'(x) = -C''(x) = C(x) \quad \Rightarrow \quad S'(x) = C(x).$$

(d) Let us now recall that Abel's Theorem says that the Wronskian of two solutions to a second order differential equation $y'' + p(x)\, y' + q(x)\, y = 0$ satisfies the differential equation $W' + p(x)\, W = 0$. In our case the function $p = 0$, so the Wronskian is a constant function. If we compute the Wronskian of the functions C and S and we use the differential relations found in (c) we get

$$W(x) = C(x)\, S'(x) - C'(x)\, S(x) = C^2(x) + S^2(x).$$

This Wronskian must be a constant function, but at $x = 0$ takes the value $W(0) = C^2(0) + S^2(0) = 1$. We therefore conclude that for all $x \in \mathbb{R}$ holds

$$C^2(x) + S^2(x) = 1.$$

(e) We end computing power series expansions of these functions C and S, so we have a way to compute their values. We start with function C. The initial conditions say

$$C(0) = 1, \quad C'(0) = 0.$$

The differential equation at $x = 0$ and the first initial condition say that $C''(0) = -C(0) = -1$. The derivative of the differential equation at $x = 0$ and the second initial condition say that $C'''(0) = -C'(0) = 0$. If we keep taking derivatives of the differential equation we get

$$C''(0) = -1, \quad C'''(0) = 0, \quad C^{(4)}(0) = 1,$$

and in general,

$$C^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^k & \text{if } n = 2k, \text{ where } k = 0, 1, 2, \cdots . \end{cases}$$

So we obtain the Taylor series expansion

$$C(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

which is the power series expansion of the cosine function. A similar calculation yields

$$S(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

which is the power series expansion of the sine function. Notice that we have obtained these expansions using only the differential equation and its derivatives at $x = 0$ together with the initial conditions. The ratio test shows that these power series converge for all $x \in \mathbb{R}$. These power series expansions also say that the function S is odd and C is even. ◁

Review of Natural Logarithms and Exponentials. The discovery, or invention, of a new type of functions happened many times before the time of differential equations. Looking at the history of mathematics we see that people first defined polynomials as additions and multiplications on the independent variable x. After that came quotient of polynomials. Then people defined trigonometric functions as ratios of geometric objects. For example the sine and cosine functions were originally defined as ratios of the sides of right triangles. These were all the functions known before calculus, before the seventeen century. Calculus brought the natural logarithm and its inverse, the exponential function together with the number e.

What is used to define the natural logarithm is not a differential equation but integration. People knew that the antiderivative of a power function $f(x) = x^n$ is another power function $F(x) = x^{(n+1)}/(n+1)$, except for $n = -1$, where this rule fails. The antiderivative of the function $f(x) = 1/x$ is neither a power function nor a trigonometric function, so at that time it was a new function. People gave a name to this new function, ln, and defined it as whatever comes from the integration of the function $f(x) = 1/x$, that is,

$$\ln(x) = \int_1^x \frac{ds}{s}, \qquad x > 0.$$

All the properties of this new function must come from that definition. It is clear that this function is increasing, that $\ln(1) = 0$, and that the function take values in $(-\infty, \infty)$. But this function has a more profound property, $\ln(ab) = \ln(a) + \ln(b)$. To see this relation first compute

$$\ln(ab) = \int_1^{ab} \frac{ds}{s} = \int_1^a \frac{ds}{s} + \int_a^{ab} \frac{ds}{s};$$

then change the variable in the second term, $\tilde{s} = s/a$, so $d\tilde{s} = ds/a$, hence $ds/s = d\tilde{s}/\tilde{s}$, and

$$\ln(ab) = \int_1^a \frac{ds}{s} + \int_1^b \frac{d\tilde{s}}{\tilde{s}} = \ln(a) + \ln(b).$$

The Euler number e is defined as the solution of the equation $\ln(e) = 1$. The inverse of the natural logarithm, $\ln^{-1}$, is defined in the usual way,

$$\ln^{-1}(y) = x \quad \Leftrightarrow \quad \ln(x) = y, \qquad x \in (0, \infty), \quad y \in (-\infty, \infty).$$

Since the natural logarithm satisfies that $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$, the inverse function satisfies the related identity $\ln^{-1}(y_1 + y_2) = \ln^{-1}(y_1)\, \ln^{-1}(y_2)$. To see this identity compute

$$\ln^{-1}(y_1 + y_2) = \ln^{-1}\big(\ln(x_1) + \ln(x_2)\big) = \ln^{-1}(\ln(x_1 x_2)) = x_1 x_2 = \ln^{-1}(y_1)\, \ln^{-1}(y_2).$$

This identity and the fact that $\ln^{-1}(1) = e$ imply that for any positive integer n holds

$$\ln^{-1}(n) = \ln^{-1}\,(\overbrace{1 + \cdots + 1}^{\text{n times}}) = \overbrace{\ln^{-1}(1) \cdots \ln^{-1}(1)}^{\text{n times}} = \overbrace{e \cdots e}^{\text{n times}} = e^n.$$

This relation says that $\ln^{-1}$ is the exponential function when restricted to positive integers. This suggests a way to generalize the exponential function from positive integers to real numbers, $e^y = \ln^{-1}(y)$, for y real. Hence the name exponential for the inverse of the natural logarithm. And this is how calculus brought us the logarithm and the exponential functions.

Finally notice that by the definition of the natural logarithm, its derivative is $\ln'(x) = 1/x$. But there is a formula relating the derivative of a function f and its inverse f^{-1},

$$\big(f^{-1}\big)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}.$$

Using this formula for the natural logarithm we see that

$$\big(\ln^{-1}\big)'(y) = \frac{1}{\ln'\big(\ln^{-1}(y)\big)} = \ln^{-1}(y).$$

In other words, the inverse of the natural logarithm, call it now $g(y) = \ln^{-1}(y) = e^{y}$, must be a solution to the differential equation

$$g'(y) = g(y).$$

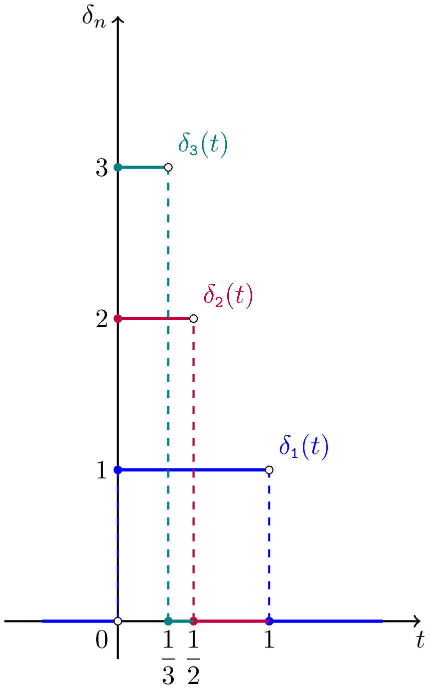
And this is how logarithms and exponentials can be added to the set of known functions. Of course, now that we know about differential equations, we can always start with the differential equation above and obtain all the properties of the exponential function using the differential equation. This might be a nice exercise for the interested reader.

CHAPTER 4

The Laplace Transform Method

The Laplace Transform is a transformation, meaning that it changes a function into a new function. Actually, it is a linear transformation, because it converts a linear combination of functions into a linear combination of the transformed functions. Even more interesting, the Laplace Transform converts derivatives into multiplications. These two properties make the Laplace Transform very useful to solve linear differential equations with constant coefficients. The Laplace Transform converts such differential equation for an unknown function into an algebraic equation for the transformed function. Usually it is easy to solve the algebraic equation for the transformed function. Then one converts the transformed function back into the original function. This function is the solution of the differential equation.

Solving a differential equation using a Laplace Transform is radically different from all the methods we have used so far. This method, as we will use it here, is relatively new. The Laplace Transform we define here was first used in 1910, but its use grew rapidly after 1920, specially to solve differential equations. Transformations like the Laplace Transform were known much earlier. Pierre Simon de Laplace used a similar transformation in his studies of probability theory, published in 1812, but analogous transformations were used even earlier by Euler around 1737.


δ_n
$\delta_3(t)$
$\delta_2(t)$
$\delta_1(t)$
3
2
1
0
$\frac{1}{3}$
$\frac{1}{2}$
1
t

4.1. Introduction to the Laplace Transform

The Laplace transform is a transformation—it changes a function into another function. This transformation is an integral transformation—the original function is multiplied by an exponential and integrated on an appropriate region. Such an integral transformation is the answer to very interesting questions: Is it possible to transform a differential equation into an algebraic equation? Is it possible to transform a derivative of a function into a multiplication? The answer to both questions is yes, for example with a Laplace transform.

This is how it works. You start with a derivative of a function, $y'(t)$, then you multiply it by any function, we choose an exponential e^{-st}, and then you integrate on t, so we get

$$y'(t) \to \int e^{-st}\, y'(t)\, dt,$$

which is a transformation, an integral transformation. And now, because we have an integration above, we can integrate by parts—this is the big idea,

$$y'(t) \to \int e^{-st}\, y'(t)\, dt = e^{-st}\, y(t) + s \int e^{-st}\, y(t)\, dt.$$

So we have transformed the derivative we started with into a multiplication by this constant s from the exponential. The idea in this calculation actually works to solve differential equations and motivates us to define the integral transformation $y(t) \to \tilde{Y}(s)$ as follows,

$$y(t) \to \tilde{Y}(s) = \int e^{-st}\, y(t)\, dt.$$

The Laplace transform is a transformation similar to the one above, where we choose some appropriate integration limits—which are very convenient to solve initial value problems.

We dedicate this section to introduce the precise definition of the Laplace transform and how is used to solve differential equations. In the following sections we will see that this method can be used to solve linear constant coefficients differential equation with very general sources, including Dirac's delta generalized functions.

4.1.1. Oveview of the Method.

The Laplace transform changes a function into another function. For example, we will show later on that the Laplace transform changes

$$f(x) = \sin(ax) \quad \text{into} \quad F(x) = \frac{a}{x^2 + a^2}.$$

We will follow the notation used in the literature and we use t for the variable of the original function f, while we use s for the variable of the transformed function F. Using this notation, the Laplace transform changes

$$f(t) = \sin(at) \quad \text{into} \quad F(s) = \frac{a}{s^2 + a^2}.$$

We will show that the Laplace transform is a linear transformation and it transforms derivatives into multiplication. Because of these properties we will use the Laplace transform to solve linear differential equations.

We Laplace transform the original differential equation. Because the the properties above, the result will be an algebraic equation for the transformed function. Algebraic equations are simple to solve, so we solve the algebraic equation. Then we Laplace transform back the solution. We summarize these steps as follows,

$$\mathcal{L}\begin{bmatrix}\text{differential}\\ \text{eq. for } y.\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic}\\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic}\\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y.\\ \text{(Use the table.)}\end{matrix}$$

4.1.2. The Laplace Transform. The Laplace transform is a transformation, meaning that it converts a function into a new function. We have seen transformations earlier in these notes. In Chapter 2 we used the transformation

$$L[y(t)] = y''(t) + a_1\, y'(t) + a_0\, y(t),$$

so that a second order linear differential equation with source f could be written as $L[y] = f$. There are simpler transformations, for example the differentiation operation itself,

$$D[f(t)] = f'(t).$$

Not all transformations involve differentiation. There are integral transformations, for example integration itself,

$$I[f(t)] = \int_0^x f(t)\, dt.$$

Of particular importance in many applications are integral transformations of the form

$$T[f(t)] = \int_a^b K(s,t)\, f(t)\, dt,$$

where K is a fixed function of two variables, called the *kernel* of the transformation, and a, b are real numbers or $\pm\infty$. The Laplace transform is a transfomation of this type, where the kernel is $K(s,t) = e^{-st}$, the constant $a = 0$, and $b = \infty$.

Definition 4.1.1. *The* ***Laplace transform*** *of a function f defined on $D_f = (0,\infty)$ is*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{4.1.1}$$

defined for all $s \in D_F \subset \mathbb{R}$ where the integral converges.

In these note we use an alternative notation for the Laplace transform that emphasizes that the Laplace transform is a transformation: $\mathcal{L}[f] = F$, that is

$$\mathcal{L}[\quad] = \int_0^\infty e^{-st}\,(\quad)\, dt.$$

So, the Laplace transform will be denoted as either $\mathcal{L}[f]$ or F, depending whether we want to emphasize the transformation itself or the result of the transformation. We will also use the notation $\mathcal{L}[f(t)]$, or $\mathcal{L}[f](s)$, or $\mathcal{L}[f(t)](s)$, whenever the independent variables t and s are relevant in any particular context.

The Laplace transform is an improper integral—an integral on an unbounded domain. Improper integrals are defined as a limit of definite integrals,

$$\int_{t_0}^\infty g(t)\, dt = \lim_{N\to\infty} \int_{t_0}^N g(t)\, dt.$$

An improper integral *converges* iff the limit exists, otherwise the integral *diverges*.

Now we are ready to compute our first Laplace transform.

Example 4.1.1. Compute the Laplace transform of the function $f(t) = 1$, that is, $\mathcal{L}[1]$.

Solution: Following the definition,

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt = \lim_{N\to\infty} \int_0^N e^{-st}\, dt.$$

The definite integral above is simple to compute, but it depends on the values of s. For $s = 0$ we get

$$\lim_{N\to\infty} \int_0^N dt = \lim_{n\to\infty} N = \infty.$$

So, the improper integral diverges for $s = 0$. For $s \neq 0$ we get

$$\lim_{N\to\infty} \int_0^N e^{-st}\,dt = \lim_{N\to\infty} -\frac{1}{s}\,e^{-st}\Big|_0^N = \lim_{N\to\infty} -\frac{1}{s}\,(e^{-sN} - 1).$$

For $s < 0$ we have $s = -|s|$, hence

$$\lim_{N\to\infty} -\frac{1}{s}\,(e^{-sN} - 1) = \lim_{N\to\infty} -\frac{1}{s}\,(e^{|s|N} - 1) = -\infty.$$

So, the improper integral diverges for $s < 0$. In the case that $s > 0$ we get

$$\lim_{N\to\infty} -\frac{1}{s}\,(e^{-sN} - 1) = \frac{1}{s}.$$

If we put all these result together we get

$$\mathcal{L}[1] = \frac{1}{s}, \qquad s > 0.$$

$\triangleleft$

Example 4.1.2. Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: We start with the definition of the Laplace transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt.$$

In the case $s = a$ we get

$$\mathcal{L}[e^{at}] = \int_0^\infty 1\,dt = \infty,$$

so the improper integral diverges. In the case $s \neq a$ we get

$$\begin{aligned} \mathcal{L}[e^{at}] &= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt, \qquad s \neq a, \\ &= \lim_{N\to\infty} \Big[\frac{(-1)}{(s-a)}\,e^{-(s-a)t}\Big|_0^N\Big] \\ &= \lim_{N\to\infty} \Big[\frac{(-1)}{(s-a)}\,(e^{-(s-a)N} - 1)\Big]. \end{aligned}$$

Now we have to remaining cases. The first case is:

$$s - a < 0 \quad\Rightarrow\quad -(s-a) = |s-a| > 0 \quad\Rightarrow\quad \lim_{N\to\infty} e^{-(s-a)N} = \infty,$$

so the integral diverges for $s < a$. The other case is:

$$s - a > 0 \quad\Rightarrow\quad -(s-a) = -|s-a| < 0 \quad\Rightarrow\quad \lim_{N\to\infty} e^{-(s-a)N} = 0,$$

so the integral converges only for $s > a$ and the Laplace transform is given by

$$\mathcal{L}[e^{at}] = \frac{1}{(s-a)}, \qquad s > a.$$

$\triangleleft$

Example 4.1.3. Compute $\mathcal{L}[te^{at}]$, where $a \in \mathbb{R}$.

Solution: In this case the calculation is more complicated than above, since we need to integrate by parts. We start with the definition of the Laplace transform,

$$\mathcal{L}[te^{at}] = \int_0^\infty e^{-st} te^{at}\,dt = \lim_{N\to\infty} \int_0^N te^{-(s-a)t}\,dt.$$

This improper integral diverges for $s = a$, so $\mathcal{L}[te^{at}]$ is not defined for $s = a$. From now on we consider only the case $s \neq a$. In this case we can integrate by parts,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N + \frac{1}{s-a}\int_0^N e^{-(s-a)t}\,dt\Big],$$

that is,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N - \frac{1}{(s-a)^2} e^{-(s-a)t}\Big|_0^N\Big]. \tag{4.1.2}$$

In the case that $s < a$ the first term above diverges,

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\,e^{-(s-a)N} = \lim_{N\to\infty} -\frac{1}{(s-a)} N\,e^{|s-a|N} = \infty,$$

therefore $\mathcal{L}[te^{at}]$ is not defined for $s < a$. In the case $s > a$ the first term on the right hand side in (4.1.2) vanishes, since

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\,e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)} t\,e^{-(s-a)t}\big|_{t=0} = 0.$$

Regarding the other term, and recalling that $s > a$,

$$\lim_{N\to\infty} -\frac{1}{(s-a)^2} e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)^2} e^{-(s-a)t}\big|_{t=0} = \frac{1}{(s-a)^2}.$$

Therefore, we conclude that

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}, \qquad s > a.$$

◁

Example 4.1.4. Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \int_0^\infty e^{-st}\sin(at)\,dt = \lim_{N\to\infty}\int_0^N e^{-st}\sin(at)\,dt$$

The definite integral above can be computed integrating by parts twice,

$$\int_0^N e^{-st}\sin(at)\,dt = -\frac{1}{s}\big[e^{-st}\sin(at)\big]\Big|_0^N - \frac{a}{s^2}\big[e^{-st}\cos(at)\big]\Big|_0^N - \frac{a^2}{s^2}\int_0^N e^{-st}\sin(at)\,dt,$$

which implies that

$$\Big(1 + \frac{a^2}{s^2}\Big)\int_0^N e^{-st}\sin(at)\,dt = -\frac{1}{s}\big[e^{-st}\sin(at)\big]\Big|_0^N - \frac{a}{s^2}\big[e^{-st}\cos(at)\big]\Big|_0^N.$$

then we get

$$\int_0^N e^{-st}\sin(at)\,dt = \frac{s^2}{(s^2+a^2)}\Big[-\frac{1}{s}\big[e^{-st}\sin(at)\big]\Big|_0^N - \frac{a}{s^2}\big[e^{-st}\cos(at)\big]\Big|_0^N\Big].$$

and finally we get

$$\int_0^N e^{-st}\sin(at)\,dt = \frac{s^2}{(s^2+a^2)}\Big[-\frac{1}{s}\big[e^{-sN}\sin(aN) - 0\big] - \frac{a}{s^2}\big[e^{-sN}\cos(aN) - 1\big]\Big].$$

One can check that the limit $N \to \infty$ on the right hand side above does not exist for $s \leqslant 0$, so $\mathcal{L}[\sin(at)]$ does not exist for $s \leqslant 0$. In the case $s > 0$ it is not difficult to see that

$$\int_0^\infty e^{-st} \sin(at)\, dt = \left(\frac{s^2}{s^2 + a^2}\right) \left[\frac{1}{s}\,(0-0) - \frac{a}{s^2}\,(0-1)\right]$$

so we obtain the final result

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

$\lhd$

In Table 1 we present a short list of Laplace transforms. They can be computed in the same way we computed the the Laplace transforms in the examples above.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	D_F
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \lvert a \rvert$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \lvert a \rvert$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \lvert b \rvert$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \lvert b \rvert$

TABLE 1. List of a few Laplace transforms.

4.1.3. Main Properties.

Since we are more or less confident on how to compute a Laplace transform, we can start asking deeper questions. For example, what type of functions have a Laplace transform? It turns out that a large class of functions, those that are piecewise continuous on $[0,\infty)$ and bounded by an exponential. This last property is particularly important and we give it a name.

Definition 4.1.2. *A function f defined on $[0,\infty)$ is of* ***exponential order*** s_0*, where s_0 is any real number, iff there exist positive constants k, T such that*

$$|f(t)| \leqslant k\, e^{s_0 t} \quad \text{for all} \quad t > T. \tag{4.1.3}$$

Remarks:

(a) When the precise value of the constant s_0 is not important we will say that f is of exponential order.

(b) An example of a function that is not of exponential order is $f(t) = e^{t^2}$.

This definition helps to describe a set of functions having Laplace transform. Piecewise continuous functions on $[0,\infty)$ of exponential order have Laplace transforms.

Theorem 4.1.3 (Convergence of LT)**.** *If a function f defined on $[0,\infty)$ is piecewise continuous and of exponential order s_0, then the $\mathcal{L}[f]$ exists for all $s > s_0$ and there exists a positive constant k such that*

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s-s_0}, \qquad s > s_0.$$

Proof of Theorem 4.1.3: From the definition of the Laplace transform we know that

$$\mathcal{L}[f] = \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt.$$

The definite integral on the interval $[0,N]$ exists for every $N > 0$ since f is piecewise continuous on that interval, no matter how large N is. We only need to check whether the integral converges as $N \to \infty$. This is the case for functions of exponential order, because

$$\Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant \int_0^N e^{-st} |f(t)|\, dt \leqslant \int_0^N e^{-st} k e^{s_0 t}\, dt = k \int_0^N e^{-(s-s_0)t}\, dt.$$

Therefore, for $s > s_0$ we can take the limit as $N \to \infty$,

$$\big|\mathcal{L}[f]\big| \leqslant \lim_{N\to\infty} \Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant k\, \mathcal{L}[e^{s_0 t}] = \frac{k}{(s-s_0)}.$$

Therefore, the comparison test for improper integrals implies that the Laplace transform $\mathcal{L}[f]$ exists at least for $s > s_0$, and it also holds that

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s-s_0}, \qquad s > s_0.$$

This establishes the Theorem. □

The next result says that the Laplace transform is a linear transformation. This means that the Laplace transform of a linear combination of functions is the linear combination of their Laplace transforms.

Theorem 4.1.4 (Linearity)**.** *If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all a, $b \in \mathbb{R}$ holds*

$$\mathcal{L}[af + bg] = a\,\mathcal{L}[f] + b\,\mathcal{L}[g].$$

Proof of Theorem 4.1.4: Since integration is a linear operation, so is the Laplace transform, as this calculation shows,

$$\begin{aligned}\mathcal{L}[af + bg] &= \int_0^\infty e^{-st}\big[af(t) + bg(t)\big]\,dt\\ &= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt\\ &= a\,\mathcal{L}[f] + b\,\mathcal{L}[g].\end{aligned}$$

This establishes the Theorem. □

Example 4.1.5. Compute $\mathcal{L}[3t^2 + 5\cos(4t)]$.

Solution: From the Theorem above and the Laplace transform in Table **??** we know that

$$\begin{aligned}\mathcal{L}[3t^2 + 5\cos(4t)] &= 3\,\mathcal{L}[t^2] + 5\,\mathcal{L}[\cos(4t)]\\ &= 3\Big(\frac{2}{s^3}\Big) + 5\Big(\frac{s}{s^2+4^2}\Big), \qquad s > 0\\ &= \frac{6}{s^3} + \frac{5s}{s^2+4^2}.\end{aligned}$$

Therefore,

$$\mathcal{L}[3t^2 + 5\cos(4t)] = \frac{5s^4 + 6s^2 + 96}{s^3(s^2+16)}, \qquad s > 0.$$

◁

The Laplace transform can be used to solve differential equations. The Laplace transform converts a differential equation into an algebraic equation. This is so because the Laplace transform converts derivatives into multiplications. Here is the precise result.

Theorem 4.1.5 (Derivative into Multiplication)**.** *If a function f is continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f']$ exists for $s > s_0$ and*

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0. \tag{4.1.4}$$

Proof of Theorem 4.1.5: The main calculation in this proof is to compute

$$\mathcal{L}[f'] = \lim_{N\to\infty}\int_0^N e^{-st} f'(t)\,dt.$$

We start computing the definite integral above. Since f' is continuous on $[0,\infty)$, that definite integral exists for all positive N, and we can integrate by parts,

$$\begin{aligned}\int_0^N e^{-st} f'(t)\,dt &= \Big[\Big(e^{-st} f(t)\Big)\Big|_0^N - \int_0^N (-s)e^{-st} f(t)\,dt\Big]\\ &= e^{-sN} f(N) - f(0) + s\int_0^N e^{-st} f(t)\,dt.\end{aligned}$$

We now compute the limit of this expression above as $N \to \infty$. Since f is continuous on $[0,\infty)$ of exponential order s_0, we know that

$$\lim_{N\to\infty}\int_0^N e^{-st} f(t)\,dt = \mathcal{L}[f], \qquad s > s_0.$$

Let us use one more time that f is of exponential order s_0. This means that there exist positive constants k and T such that $|f(t)| \leqslant k\, e^{s_0 t}$, for $t > T$. Therefore,

$$\lim_{N\to\infty} e^{-sN} f(N) \leqslant \lim_{N\to\infty} k\, e^{-sN} e^{s_0 N} = \lim_{N\to\infty} k\, e^{-(s-s_0)N} = 0, \qquad s > s_0.$$

These two results together imply that $\mathcal{L}[f']$ exists and holds

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0.$$

This establishes the Theorem. $\square$

Example 4.1.6. Verify the result in Theorem 4.1.5 for the function $f(t) = \cos(bt)$.

Solution: We need to compute the left hand side and the right hand side of Eq. (4.1.4) and verify that we get the same result. We start with the left hand side,

$$\mathcal{L}[f'] = \mathcal{L}[-b\sin(bt)] = -b\,\mathcal{L}[\sin(bt)] = -b\,\frac{b}{s^2+b^2} \quad\Rightarrow\quad \mathcal{L}[f'] = -\frac{b^2}{s^2+b^2}.$$

We now compute the right hand side,

$$s\,\mathcal{L}[f] - f(0) = s\,\mathcal{L}[\cos(bt)] - 1 = s\,\frac{s}{s^2+b^2} - 1 = \frac{s^2 - s^2 - b^2}{s^2+b^2},$$

so we get

$$s\,\mathcal{L}[f] - f(0) = -\frac{b^2}{s^2+b^2}.$$

We conclude that $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$. $\triangleleft$

It is not difficult to generalize Theorem 4.1.5 to higher order derivatives.

Theorem 4.1.6 (Higher Derivatives into Multiplication)**.** *If a function f is n-times continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f''], \cdots, \mathcal{L}[f^{(n)}]$ exist for $s > s_0$ and*

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) \tag{4.1.5}$$

$$\vdots$$

$$\mathcal{L}[f^{(n)}] = s^n\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0). \tag{4.1.6}$$

Proof of Theorem 4.1.6: We need to use Eq. (4.1.4) n times. We start with the Laplace transform of a second derivative,

$$\begin{aligned}\mathcal{L}[f''] &= \mathcal{L}[(f')'] \\ &= s\,\mathcal{L}[f'] - f'(0) \\ &= s\big(s\,\mathcal{L}[f] - f(0)\big) - f'(0) \\ &= s^2\,\mathcal{L}[f] - s\,f(0) - f'(0).\end{aligned}$$

The formula for the Laplace transform of an nth derivative is computed by induction on n. We assume that the formula is true for $n-1$,

$$\mathcal{L}[f^{(n-1)}] = s^{(n-1)}\,\mathcal{L}[f] - s^{(n-2)}\,f(0) - \cdots - f^{(n-2)}(0).$$

Since $\mathcal{L}[f^{(n)}] = \mathcal{L}[(f')^{(n-1)}]$, the formula above on f' gives

$$\begin{aligned}\mathcal{L}[(f')^{(n-1)}] &= s^{(n-1)}\,\mathcal{L}[f'] - s^{(n-2)}\,f'(0) - \cdots - (f')^{(n-2)}(0)\\ &= s^{(n-1)}\,\big(s\,\mathcal{L}[f] - f(0)\big) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0)\\ &= s^{(n)}\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0).\end{aligned}$$

This establishes the Theorem. □

Example 4.1.7. Verify Theorem 4.1.6 for f'', where $f(t) = \cos(bt)$.

Solution: We need to compute the left hand side and the right hand side in the first equation in Theorem (4.1.6), and verify that we get the same result. We start with the left hand side,

$$\mathcal{L}[f''] = \mathcal{L}[-b^2\cos(bt)] = -b^2\,\mathcal{L}[\cos(bt)] = -b^2\,\frac{s}{s^2+b^2} \quad\Rightarrow\quad \mathcal{L}[f''] = -\frac{b^2 s}{s^2+b^2}.$$

We now compute the right hand side,

$$s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) = s^2\,\mathcal{L}[\cos(bt)] - s - 0 = s^2\,\frac{s}{s^2+b^2} - s = \frac{s^3 - s^3 - b^2 s}{s^2+b^2},$$

so we get

$$s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) = -\frac{b^2 s}{s^2+b^2}.$$

We conclude that $\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0)$. ◁

The Laplace transform also satisfies a converse to Theorem 4.1.5, since multiplications can be transformed into derivatives.

Theorem 4.1.7 (Multiplication into Derivative)**.** *If a function f is of exponential order s_0 with a Laplace transform $F(s) = \mathcal{L}[f(t)]$, then $\mathcal{L}[t\,f(t)]$ exists for $s > s_0$ and*

$$\mathcal{L}[t\,f(t)] = -F'(s), \qquad s > s_0. \tag{4.1.7}$$

Proof of Theorem 4.1.7: From the definition of the Laplace Transform we see that

$$\begin{aligned}\mathcal{L}[t\,f(t)] &= \int_0^\infty e^{-st}\,t\,f(t)\,dt\\ &= \int_0^\infty \frac{d}{ds}\Big(-e^{-st}\Big)\,f(t)\,dt\\ &= -\frac{d}{ds}\int_0^\infty e^{-st}\,f(t)\,dt\\ &= -\frac{d}{ds}\mathcal{L}[f(t)]\\ &= -F'(s).\end{aligned}$$

This establishes the Theorem. □

The result in Theorem 4.1.7 can be generalized to higher powers.

Theorem 4.1.8 (Higher Powers into Derivative)**.** *If a function f is of exponential order s_0 with a Laplace transform $F(s) = \mathcal{L}[f(t)]$, then $\mathcal{L}[t^n\,f(t)]$ exists for $s > s_0$ and*

$$\mathcal{L}[t^n\,f(t)] = (-1)^n F^{(n)}(s), \qquad s > s_0, \tag{4.1.8}$$

where we denoted $F^{(n)} = \dfrac{d^n}{ds^n}F$.

Proof of Theorem 4.1.8: We use induction one more time. The case $n = 1$ is done in Theorem 4.1.7. We now assume that

$$\mathcal{L}[t^n\, f(t)] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)],$$

and we try to show that a similar formula holds for $n+1$. But this is the case, since

$$\begin{aligned}
\mathcal{L}[t^{(n+1)}\, f(t)] &= \mathcal{L}\big[t^n\big(t\, f(t)\big)\big]\\
&= (-1)^n \frac{d^n}{ds^n}\mathcal{L}[t\, f(t)],
\end{aligned}$$

since $t\, f(t)$ satisfies the hypotheses in Theorem 4.1.7, since $f(t)$ does. Then we use Theorem 4.1.7 one more time,

$$\begin{aligned}
\mathcal{L}[t^{(n+1)}\, f(t)] &= (-1)^n \frac{d^n}{ds^n}\mathcal{L}[t\, f(t)],\\
&= (-1)^n \frac{d^n}{ds^n}(-1)\frac{d}{ds}\mathcal{L}[f(t)],\\
&= (-1)^{(n+1)} \frac{d^{(n+1)}}{ds^{(n+1)}}\mathcal{L}[f(t)],\\
&= (-1)^{(n+1)} F^{(n+1)}(s).
\end{aligned}$$

This establishes the Theorem. □

4.1.4. Solving Differential Equations.

The Laplace transform can be used to solve differential equations. We Laplace transform the whole equation, which converts the differential equation for y into an algebraic equation for $\mathcal{L}[y]$. We solve the Algebraic equation and we transform back.

$$\mathcal{L}\begin{bmatrix}\text{differential}\\ \text{eq. for } y.\end{bmatrix} \xrightarrow{(1)} \begin{array}{c}\text{Algebraic}\\ \text{eq. for } \mathcal{L}[y].\end{array} \xrightarrow{(2)} \begin{array}{c}\text{Solve the}\\ \text{algebraic}\\ \text{eq. for } \mathcal{L}[y].\end{array} \xrightarrow{(3)} \begin{array}{c}\text{Transform back}\\ \text{to obtain } y.\\ \text{(Use the table.)}\end{array}$$

Example 4.1.8. Use the Laplace transform to find y solution of

$$y'' + 9\, y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

Remark: Notice we already know what the solution of this problem is. Following § 2.3 we need to find the roots of

$$p(r) = r^2 + 9 \quad \Rightarrow \quad r_\pm = \pm 3\, i,$$

and then we get the general solution

$$y(t) = c_+ \cos(3t) + c_- \sin(3t).$$

Then the initial condition will say that

$$y(t) = y_0\, \cos(3t) + \frac{y_1}{3}\, \sin(3t).$$

We now solve this problem using the Laplace transform method.

Solution: We now use the Laplace transform method:

$$\mathcal{L}[y'' + 9y] = \mathcal{L}[0] = 0.$$

The Laplace transform is a linear transformation,

$$\mathcal{L}[y''] + 9\, \mathcal{L}[y] = 0.$$

But the Laplace transform converts derivatives into multiplications,

$$s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) + 9\,\mathcal{L}[y] = 0.$$

This is an algebraic equation for $\mathcal{L}[y]$. It can be solved by rearranging terms and using the initial condition,

$$(s^2+9)\,\mathcal{L}[y] = s\,y_0 + y_1 \quad \Rightarrow \quad \mathcal{L}[y] = y_0\,\frac{s}{(s^2+9)} + y_1\,\frac{1}{(s^2+9)}.$$

But from the Laplace transform table we see that

$$\mathcal{L}[\cos(3t)] = \frac{s}{s^2+3^2}, \qquad \mathcal{L}[\sin(3t)] = \frac{3}{s^2+3^2},$$

therefore,

$$\mathcal{L}[y] = y_0\,\mathcal{L}[\cos(3t)] + y_1\,\frac{1}{3}\,\mathcal{L}[\sin(3t)].$$

Once again, the Laplace transform is a linear transformation,

$$\mathcal{L}[y] = \mathcal{L}\Big[y_0\,\cos(3t) + \frac{y_1}{3}\,\sin(3t)\Big].$$

We obtain that

$$y(t) = y_0\,\cos(3t) + \frac{y_1}{3}\,\sin(3t).$$

$\triangleleft$

4.1.5. Exercises.

4.1.1.-

(a) Compute the definite integral

$$I_N = \int_4^N \frac{dx}{x^2}.$$

(b) Use the result in (a) to compute

$$I = \int_4^\infty \frac{dx}{x^2}.$$

4.1.2.-

(a) Compute the definite integral

$$I_N = \int_5^N e^{-st}\, dt.$$

(b) Use the result in (a) to compute

$$I = \int_5^\infty e^{-st}\, dt.$$

4.1.3.-

(a) Compute the definite integral

$$I_N = \int_0^N e^{-st}\, e^{2t}\, dt.$$

(b) Use the result in (a) to compute

$$F(s) = \mathcal{L}[e^{2t}].$$

Indicate the domain of F.

4.1.4.-

(a) Compute the definite integral

$$I_N = \int_0^N e^{-st}\, t\, e^{-2t}\, dt.$$

(b) Use the result in (a) to compute

$$F(s) = \mathcal{L}[t\, e^{-2t}].$$

Indicate the domain of F.

4.1.5.-

(a) Compute the definite integral

$$I_N = \int_0^N e^{-st}\, \sin(2t)\, dt.$$

(b) Use the result in (a) to compute

$$F(s) = \mathcal{L}[\sin(2t)].$$

Indicate the domain of F.

4.1.6.-

(a) Compute the definite integral

$$I_N = \int_0^N e^{-st}\, \cos(2t)\, dt.$$

(b) Use the result in (a) to compute

$$F(s) = \mathcal{L}[\cos(2t)].$$

Indicate the domain of F.

4.1.7.- Use the definition of the Laplace transform to compute

$$F(s) = \mathcal{L}[\sinh(at)],$$

and indicate the domain of F.

4.1.8.- ***** Use the definition of the Laplace transform to compute

$$F(s) = \mathcal{L}[\cosh(at)],$$

and indicate the domain of F.

4.2. The Initial Value Problem

We will use the Laplace transform to solve differential equations. The main idea is,

$$\mathcal{L}\begin{bmatrix}\text{differential eq.}\\ \text{for } y(t).\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Use the table.)}\end{matrix}$$

We will use the Laplace transform to solve differential equations with *constant coefficients*. Although the method can be used with variable coefficients equations, the calculations could be very complicated in such a case.

The Laplace transform method works with *very general source functions*, including step functions, which are discontinuous, and Dirac's deltas, which are generalized functions.

4.2.1. Solving Differential Equations.

As we see in the sketch above, we start with a differential equation for a function y. We first compute the Laplace transform of the whole differential equation. Then we use the linearity of the Laplace transform, Theorem 4.1.4, and the property that derivatives are converted into multiplications, Theorem 4.1.5, to transform the differential equation into an algebraic equation for $\mathcal{L}[y]$. Let us see how this works in a simple example, a first order linear equation with constant coefficients—we already solved it in § 1.1.

Example 4.2.1. Use the Laplace transform to find the solution y to the initial value problem

$$y' + 2y = 0, \qquad y(0) = 3.$$

Solution: In § 1.1 we saw one way to solve this problem, using the integrating factor method. One can check that the solution is $y(t) = 3e^{-2t}$. We now use the Laplace transform. First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0] = 0.$$

Theorem 4.1.4 says the Laplace transform is a linear operation, that is,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 0.$$

Theorem 4.1.5 relates derivatives and multiplications, as follows,

$$\Big(s\,\mathcal{L}[y] - y(0)\Big) + 2\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s+2)\mathcal{L}[y] = y(0).$$

In the last equation we have been able to transform the original differential equation for y into an algebraic equation for $\mathcal{L}[y]$. We can solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{y(0)}{s+2} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s+2},$$

where in the last step we introduced the initial condition $y(0) = 3$. From the list of Laplace transforms given in §. 4.1 we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{3}{s+2} = 3\mathcal{L}[e^{-2t}] \quad \Rightarrow \quad \frac{3}{s+2} = \mathcal{L}[3\,e^{-2t}].$$

So we arrive at $\mathcal{L}[y(t)] = \mathcal{L}[3\,e^{-2t}]$. Here is where we need one more property of the Laplace transform. We show right after this example that

$$\mathcal{L}[y(t)] = \mathcal{L}[3\,e^{-2t}] \quad \Rightarrow \quad y(t) = 3\,e^{-2t}.$$

This property is called one-to-one. Hence the only solution is $y(t) = 3\,e^{-2t}$. ◁

4.2.2. One-to-One Property. Let us repeat the method we used to solve the differential equation in Example 4.2.1. We first computed the Laplace transform of the whole differential equation. Then we use the linearity of the Laplace transform, Theorem 4.1.4, and the property that derivatives are converted into multiplications, Theorem 4.1.5, to transform the differential equation into an algebraic equation for $\mathcal{L}[y]$. We solved the algebraic equation and we got an expression of the form

$$\mathcal{L}[y(t)] = H(s),$$

where we have collected all the terms that come from the Laplace transformed differential equation into the function H. We then used a Laplace transform table to find a function h such that

$$\mathcal{L}[h(t)] = H(s).$$

We arrived to an equation of the form

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)].$$

Clearly, $y = h$ is one solution of the equation above, hence a solution to the differential equation. We now show that there are no solutions to the equation $\mathcal{L}[y] = \mathcal{L}[h]$ other than $y = h$. The reason is that the Laplace transform on continuous functions of exponential order is an one-to-one transformation, also called injective.

Theorem 4.2.1 (One-to-One). *If f, g are continuous on $[0, \infty)$ of exponential order, then*

$$\mathcal{L}[f] = \mathcal{L}[g] \quad \Rightarrow \quad f = g.$$

Remarks:

(a) The result above holds for continuous functions f and g. But it can be extended to piecewise continuous functions. In the case of piecewise continuous functions f and g satisfying $\mathcal{L}[f] = \mathcal{L}[g]$ one can prove that $f = g + h$, where h is a null function, meaning that $\int_0^T h(t)\,dt = 0$ for all $T > 0$. See Churchill's textbook [4], page 14.

(b) Once we know that the Laplace transform is a one-to-one transformation, we can define the inverse transformation in the usual way.

Definition 4.2.2. *The* ***inverse Laplace transform****, denoted $\mathcal{L}^{-1}$, of a function F is*

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \Leftrightarrow \quad F(s) = \mathcal{L}[f(t)].$$

Remarks: There is an explicit formula for the inverse Laplace transform, which involves an integral on the complex plane,

$$\mathcal{L}^{-1}[F(s)]\Big|_t = \frac{1}{2\pi i} \lim_{c\to\infty} \int_{a-ic}^{a+ic} e^{st}\, F(s)\, ds.$$

See for example Churchill's textbook [4], page 176. However, we do not use this formula in these notes, since it involves integration on the complex plane.

Proof of Theorem 4.2.1: The proof is based on a clever change of variables and on Weierstrass Approximation Theorem of continuous functions by polynomials. Before we get to the change of variable we need to do some rewriting. Introduce the function $u = f - g$, then the linearity of the Laplace transform implies

$$\mathcal{L}[u] = \mathcal{L}[f - g] = \mathcal{L}[f] - \mathcal{L}[g] = 0.$$

What we need to show is that the function u vanishes identically. Let us start with the definition of the Laplace transform,

$$\mathcal{L}[u] = \int_0^\infty e^{-st}\, u(t)\, dt.$$

We know that f and g are of exponential order, say s_0, therefore u is of exponential order s_0, meaning that there exist positive constants k and T such that

$$\big|u(t)\big| < k\, e^{s_0 t}, \qquad t > T.$$

Evaluate $\mathcal{L}[u]$ at $\tilde{s} = s_1 + n + 1$, where s_1 is any real number such that $s_1 > s_0$, and n is any positive integer. We get

$$\mathcal{L}[u]\Big|_{\tilde{s}} = \int_0^\infty e^{-(s_1+n+1)t}\, u(t)\, dt = \int_0^\infty e^{-s_1 t}\, e^{-(n+1)t}\, u(t)\, dt.$$

We now do the substitution $y = e^{-t}$, so $dy = -e^{-t}\, dt$,

$$\mathcal{L}[u]\Big|_{\tilde{s}} = \int_1^0 y^{s_1}\, y^n\, u\big(-\ln(y)\big)\, (-dy) = \int_0^1 y^{s_1}\, y^n\, u\big(-\ln(y)\big)\, dy.$$

Introduce the function $v(y) = y^{s_1}\, u\big((-\ln(y)\big)$, so the integral is

$$\mathcal{L}[u]\Big|_{\tilde{s}} = \int_0^1 y^n\, v(y)\, dy. \tag{4.2.1}$$

We know that $\mathcal{L}[u]$ exists because u is continuous and of exponential order, so the function v does not diverge at $y = 0$. To double check this, recall that $t = -\ln(y) \to \infty$ as $y \to 0^+$, and u is of exponential order s_0, hence

$$\lim_{y\to 0^+} |v(y)| = \lim_{t\to\infty} e^{-s_1 t}|u(t)| < \lim_{t\to\infty} e^{-(s_1-s_0)t} = 0.$$

Our main hypothesis is that $\mathcal{L}[u] = 0$ for all values of s such that $\mathcal{L}[u]$ is defined, in particular $\tilde{s}$. By looking at Eq. (4.2.1) this means that

$$\int_0^1 y^n\, v(y)\, dy = 0, \qquad n = 1, 2, 3, \cdots.$$

The equation above and the linearity of the integral imply that this function v is perpendicular to every polynomial p, that is

$$\int_0^1 p(y)\, v(y)\, dy = 0, \tag{4.2.2}$$

for every polynomial p. Knowing that, we can do the following calculation,

$$\int_0^1 v^2(y)\, dy = \int_0^1 \big(v(y) - p(y)\big)\, v(y)\, dy + \int_0^1 p(y)\, v(y)\, dy.$$

The last term in the second equation above vanishes because of Eq. (4.2.2), therefore

$$\begin{aligned}\int_0^1 v^2(y)\, dy &= \int_0^1 \big(v(y) - p(y)\big)\, v(y)\, dy\\ &\leqslant \int_0^1 \big|v(y) - p(y)\big|\, |v(y)|\, dy\\ &\leqslant \max_{y\in[0,1]} |v(y)| \int_0^1 \big|v(y) - p(y)\big|\, dy.\end{aligned} \tag{4.2.3}$$

We remark that the inequality above is true for every polynomial p. Here is where we use the Weierstrass Approximation Theorem, which essentially says that every continuous function on a closed interval can be approximated by a polynomial.

Theorem 4.2.3 (Weierstrass Approximation). *If f is a continuous function on a closed interval $[a,b]$, then for every $\epsilon > 0$ there exists a polynomial q_ϵ such that*

$$\max_{y\in[a,b]} |f(y) - q_\epsilon(y)| < \epsilon.$$

The proof of this theorem can be found on a real analysis textbook. Weierstrass result implies that, given v and $\epsilon > 0$, there exists a polynomial p_ϵ such that the inequality in (4.2.3) has the form

$$\int_0^1 v^2(y)\,dy \leqslant \max_{y\in[0,1]} |v(y)| \int_0^1 \big|v(y) - p_\epsilon(y)\big|\,dy \leqslant \max_{y\in[0,1]} |v(y)|\,\epsilon.$$

Since ϵ can be chosen as small as we please, we get

$$\int_0^1 v^2(y)\,dy = 0.$$

But v is continuous, hence $v = 0$, meaning that $f = g$. This establishes the Theorem. ☐

4.2.3. Partial Fractions.

We are now ready to start using the Laplace transform to solve second order linear differential equations with constant coefficients. The differential equation for y will be transformed into an algebraic equation for $\mathcal{L}[y]$. We will then arrive to an equation of the form $\mathcal{L}[y(t)] = H(s)$. We will see, already in the first example below, that usually this function H does not appear in Table 1. We will need to rewrite H as a linear combination of simpler functions, each one appearing in Table 1. One of the more used techniques to do that is called Partial Fractions. Let us solve the next example.

Example 4.2.2. Use the Laplace transform to find the solution y to the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] = 0.$$

Theorem 4.1.4 says that the Laplace transform is a linear operation,

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

Then, Theorem 4.1.5 relates derivatives and multiplications,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

which is equivalent to the equation

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0).$$

Once again we have transformed the original differential equation for y into an algebraic equation for $\mathcal{L}[y]$. Introduce the initial condition into the last equation above, that is,

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1).$$

Solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}.$$

The function on the right hand side above does not appear in Table 1. We now use *partial fractions* to find a function whose Laplace transform is the right hand side of the equation above. First find the roots of the polynomial in the denominator,

$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$

that is, the polynomial has two real roots. In this case we factorize the denominator,

$$\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}.$$

The partial fraction decomposition of the right-hand side in the equation above is the following: Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

A simple calculation shows

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)} = \frac{s(a+b) + (a-2b)}{(s-2)(s+1)}.$$

Hence constants a and b must be solutions of the equations

$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1. \end{cases}$$

The solution is $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Hence,

$$\mathcal{L}[y] = \frac{1}{3}\,\frac{1}{(s-2)} + \frac{2}{3}\,\frac{1}{(s+1)}.$$

From the list of Laplace transforms given in § **??**, Table 1, we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{s+1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3}\,\mathcal{L}[e^{2t}] + \frac{2}{3}\,\mathcal{L}[e^{-t}] = \mathcal{L}\Big[\frac{1}{3}\big(e^{2t} + 2\,e^{-t}\big)\Big]$$

We conclude that

$$y(t) = \frac{1}{3}\big(e^{2t} + 2\,e^{-t}\big).$$

◁

The Partial Fraction Method is usually introduced in a second course of Calculus to integrate rational functions. We need it here to use Table 1 to find Inverse Laplace transforms. The method applies to rational functions

$$R(s) = \frac{Q(s)}{P(s)},$$

where P, Q are polynomials and the degree of the numerator is less than the degree of the denominator. In the example above

$$R(s) = \frac{(s-1)}{(s^2 - s - 2)}.$$

One starts rewriting the polynomial in the denominator as a product of polynomials degree two or one. In the example above,

$$R(s) = \frac{(s-1)}{(s-2)(s+1)}.$$

One then rewrites the rational function as an addition of simpler rational functions. In the example above,

$$R(s) = \frac{a}{(s-2)} + \frac{b}{(s+1)}.$$

We now solve a few examples to recall the different partial fraction cases that can appear when solving differential equations.

Example 4.2.3. Use the Laplace transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

Theorem 4.1.4 says that the Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = 0.$$

Theorem 4.1.5 relates derivatives with multiplication,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

which is equivalent to the equation

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0).$$

Introduce the initial conditions $y(0) = 1$ and $y'(0) = 1$ into the equation above,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3.$$

Solve the algebraic equation for $\mathcal{L}[y]$,

$$\mathcal{L}[y] = \frac{(s-3)}{(s^2 - 4s + 4)}.$$

We now want to find a function y whose Laplace transform is the right hand side in the equation above. In order to see if partial fractions will be needed, we now find the roots of the polynomial in the denominator,

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-16}\big] \quad \Rightarrow \quad s_+ = s_- = 2.$$

that is, the polynomial has a single real root, so $\mathcal{L}[y]$ can be written as

$$\mathcal{L}[y] = \frac{(s-3)}{(s-2)^2}.$$

This expression is already in the partial fraction decomposition. We now rewrite the right hand side of the equation above in a way it is simple to use the Laplace transform table in § **??**,

$$\mathcal{L}[y] = \frac{(s-2) + 2 - 3}{(s-2)^2} = \frac{(s-2)}{(s-2)^2} - \frac{1}{(s-2)^2} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

From the list of Laplace transforms given in Table 1, § **??** we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2} \quad \Rightarrow \quad \frac{1}{(s-2)^2} = \mathcal{L}[t\,e^{2t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[t\,e^{2t}] = \mathcal{L}\big[e^{2t} - t\,e^{2t}\big] \quad \Rightarrow \quad y(t) = e^{2t} - t\,e^{2t}.$$

◁

Example 4.2.4. Use the Laplace transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 3\,e^{t}, \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\,e^{t}] = 3\Big(\frac{1}{s-1}\Big).$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{3}{s-1}.$$

The Laplace transform relates derivatives with multiplication,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{3}{s-1},$$

But the initial conditions are $y(0) = 0$ and $y'(0) = 0$, so

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = \frac{3}{s-1}.$$

Solve the algebraic equation for $\mathcal{L}[y]$,

$$\mathcal{L}[y] = \frac{3}{(s-1)(s^2 - 4s + 4)}.$$

We use *partial fractions* to simplify the right-hand side above. We start finding the roots of the polynomial in the denominator,

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16-16}\big] \quad \Rightarrow \quad s_+ = s_- = 2.$$

that is, the polynomial has a single real root, so $\mathcal{L}[y]$ can be written as

$$\mathcal{L}[y] = \frac{3}{(s-1)(s-2)^2}.$$

The partial fraction decomposition of the right-hand side above is

$$\frac{3}{(s-1)(s-2)^2} = \frac{a}{(s-1)} + \frac{b\,s + c}{(s-2)^2} = \frac{a\,(s-2)^2 + (b\,s + c)(s-1)}{(s-1)(s-2)^2}$$

From the far right and left expressions above we get

$$3 = a\,(s-2)^2 + (b\,s + c)(s-1) = a\,(s^2 - 4s + 4) + b\,s^2 - b\,s + c\,s - c$$

Expanding all terms above, and reordering terms, we get

$$(a + b)\,s^2 + (-4a - b + c)\,s + (4a - c - 3) = 0.$$

Since this polynomial in s vanishes for all $s \in \mathbb{R}$, we get that

$$\left.\begin{aligned} a + b &= 0, \\ -4a - b + c &= 0, \\ 4a - c - 3 &= 0. \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} a &= 3 \\ b &= -3 \\ c &= 9. \end{aligned}\right.$$

So we get

$$\mathcal{L}[y] = \frac{3}{(s-1)(s-2)^2} = \frac{3}{s-1} + \frac{-3s+9}{(s-2)^2}$$

One last trick is needed on the last term above,

$$\frac{-3s+9}{(s-2)^2} = \frac{-3(s-2+2)+9}{(s-2)^2} = \frac{-3(s-2)}{(s-2)^2} + \frac{-6+9}{(s-2)^2} = -\frac{3}{(s-2)} + \frac{3}{(s-2)^2}.$$

So we finally get

$$\mathcal{L}[y] = \frac{3}{s-1} - \frac{3}{(s-2)} + \frac{3}{(s-2)^2}.$$

From our Laplace transforms Table we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad\Rightarrow\quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2} \quad\Rightarrow\quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the formula

$$\mathcal{L}[y] = 3\,\mathcal{L}[e^t] - 3\,\mathcal{L}[e^{2t}] + 3\,\mathcal{L}[t\,e^{2t}] = \mathcal{L}\big[3\,(e^t - e^{2t} + t\,e^{2t})\big]$$

So we conclude that $y(t) = 3\,(e^t - e^{2t} + t\,e^{2t})$. ◁

Example 4.2.5. Use the Laplace transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)].$$

The right hand side above can be expressed as follows,

$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2+2^2} = \frac{6}{s^2+4}.$$

Theorem 4.1.4 says that the Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2+4},$$

and Theorem 4.1.5 relates derivatives with multiplications,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2+4}.$$

Reorder terms,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

Introduce the initial conditions $y(0) = 1$ and $y'(0) = 1$,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \frac{6}{s^2+4}.$$

Solve this algebraic equation for $\mathcal{L}[y]$, that is,

$$\mathcal{L}[y] = \frac{(s-3)}{(s^2-4s+4)} + \frac{6}{(s^2-4+4)(s^2+4)}.$$

From the Example above we know that $s^2 - 4s + 4 = (s-2)^2$, so we obtain

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}. \tag{4.2.4}$$

From the previous example we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s-2} - \frac{1}{(s-2)^2}. \tag{4.2.5}$$

We know use *partial fractions* to simplify the third term on the right hand side of Eq. (4.2.4). The appropriate partial fraction decomposition for this term is the following: Find constants a, b, c, d, such that

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}$$

Take common denominator on the right hand side above, and one obtains the system

$$\begin{aligned} a + c &= 0, \\ -4a + b - 2c + d &= 0, \\ 4a - 4b + 4c &= 0, \\ 4b - 8c + 4d &= 6. \end{aligned}$$

The solution for this linear system of equations is the following:

$$a = \frac{3}{8}, \qquad b = 0, \qquad c = -\frac{3}{8}, \qquad d = \frac{3}{4}.$$

Therefore,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\frac{s}{s^2+4} - \frac{3}{8}\,\frac{1}{(s-2)} + \frac{3}{4}\,\frac{1}{(s-2)^2}$$

We can rewrite this expression above in terms of the Laplace transforms given in Table 1, in Sect. **??**, as follows,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\mathcal{L}[\cos(2t)] - \frac{3}{8}\,\mathcal{L}[e^{2t}] + \frac{3}{4}\,\mathcal{L}[te^{2t}],$$

and using the linearity of the Laplace transform,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \mathcal{L}\Big[\frac{3}{8}\,\cos(2t) - \frac{3}{8}\,e^{2t} + \frac{3}{4}\,te^{2t}\Big]. \tag{4.2.6}$$

Finally, introducing Eqs. (4.2.5) and (4.2.6) into Eq. (4.2.4) we obtain

$$\mathcal{L}[y(t)] = \mathcal{L}\Big[(1-t)\,e^{2t} + \frac{3}{8}\,(-1+2t)\,e^{2t} + \frac{3}{8}\,\cos(2t)\Big].$$

Since the Laplace transform is an invertible transformation, we conclude that

$$y(t) = (1-t)\,e^{2t} + \frac{3}{8}\,(2t-1)\,e^{2t} + \frac{3}{8}\,\cos(2t).$$

◁

4.2.4. Higher Order IVP. The Laplace transform method can be used with linear differential equations of higher order than second order, as long as the equation coefficients are constant. Below we show how we can solve a fourth order equation.

Example 4.2.6. Use the Laplace transform to find the solution y to the initial value problem

$$y^{(4)} - 4y = 0, \qquad \begin{array}{ll} y(0) = 1, & y'(0) = 0, \\ y''(0) = -2, & y'''(0) = 0. \end{array}$$

Solution: Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y^{(4)} - 4y] = \mathcal{L}[0] = 0.$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y^{(4)}] - 4\,\mathcal{L}[y] = 0,$$

and the Laplace transform relates derivatives with multiplications,

$$\Big[s^4\,\mathcal{L}[y] - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0)\Big] - 4\,\mathcal{L}[y] = 0.$$

From the initial conditions we get

$$\Big[s^4\,\mathcal{L}[y] - s^3 - 0 + 2s - 0\Big] - 4\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\mathcal{L}[y] = s^3 - 2s \quad \Rightarrow \quad \mathcal{L}[y] = \frac{(s^3 - 2s)}{(s^4 - 4)}.$$

In this case we are lucky, because

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} = \frac{s}{(s^2 + 2)}.$$

Since

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2},$$

we get that

$$\mathcal{L}[y] = \mathcal{L}[\cos(\sqrt{2}t)] \quad \Rightarrow \quad y(t) = \cos(\sqrt{2}t).$$

$\triangleleft$

4.2.5. Exercises.

4.2.1.- .

4.2.2.- .

4.3. Discontinuous Sources

The Laplace transform method can be used to solve linear differential equations with discontinuous sources. In this section we review the simplest discontinuous function—the step function—and we use steps to construct more general piecewise continuous functions. Then, we compute the Laplace transform of a step function. But the main result in this section are the translation identities, Theorem 4.3.3. These identities, together with the Laplace transform table in § 4.1, can be very useful to solve differential equations with discontinuous sources.

4.3.1. Step Functions.

We start with a definition of a step function.

Definition 4.3.1. *The* ***step function*** *at $t = 0$ is denoted by u and given by*

$$u(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geqslant 0. \end{cases} \tag{4.3.1}$$

Example 4.3.1. Graph the step u, $u_c(t) = u(t-c)$, and $u_{-c}(t) = u(t+c)$, for $c > 0$.

Solution: The step function u and its right and left translations are plotted in Fig. 1.

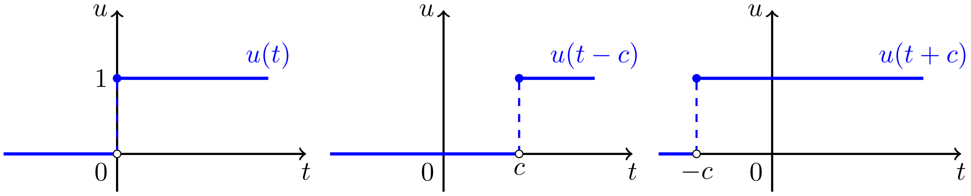


FIGURE 1. The graph of the step function given in Eq. (4.3.1), a right and a left translation by a constant $c > 0$, respectively, of this step function.

◁

Recall that given a function with values $f(t)$ and a positive constant c, then $f(t-c)$ and $f(t+c)$ are the function values of the right translation and the left translation, respectively, of the original function f. In Fig. 2 we plot the graph of functions $f(t) = e^{at}$, $g(t) = u(t)\, e^{at}$ and their respective right translations by $c > 0$.

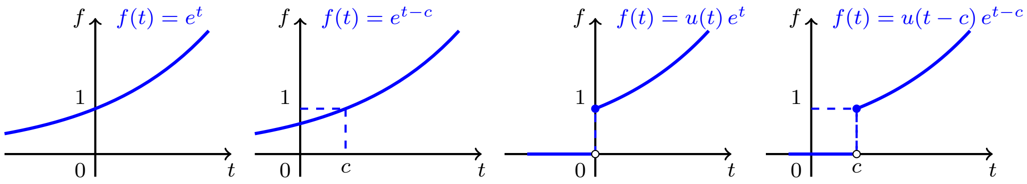


FIGURE 2. The function $f(t) = e^t$, its right translation by $c > 0$, the function $f(t) = u(t)\, e^{at}$ and its right translation by c.

Right and left translations of step functions are useful to construct bump functions.

Example 4.3.2. Graph the bump function $b(t) = u(t-a) - u(t-b)$, where $a < b$.

Solution: The bump function we need to graph is

$$b(t) = u(t-a) - u(t-b) \quad \Leftrightarrow \quad b(t) = \begin{cases} 0 & t < a, \\ 1 & a \leqslant t < b \\ 0 & t \geqslant b. \end{cases} \tag{4.3.2}$$

The graph of a bump function is given in Fig. 3, constructed from two step functions. Step and bump functions are useful to construct more general piecewise continuous functions.

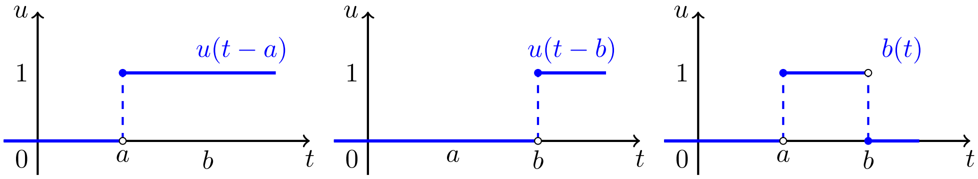


FIGURE 3. A bump function b constructed with translated step functions.

◁

Example 4.3.3. Graph the function

$$f(t) = \big[u(t-1) - u(t-2)\big]\, e^{at}.$$

Solution: Recall that the function

$$b(t) = u(t-1) - u(t-2),$$

is a bump function with sides at $t = 1$ andf $t = 2$. Then, the function

$$f(t) = b(t)\, e^{at},$$

is nonzero where b is nonzero, that is on $[1, 2)$, and on that domain it takes values e^{at}. The graph of f is given in Fig. 4. ◁

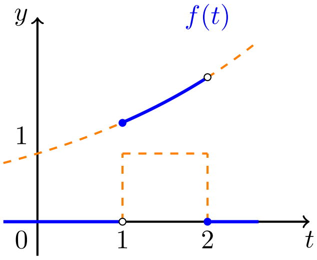


FIGURE 4. Function f.

4.3.2. The Laplace Transform of Steps. We compute the Laplace transform of a step function using the definition of the Laplace transform.

Theorem 4.3.2. *For every number $c \in \mathbb{R}$ and and every $s > 0$ holds*

$$\mathcal{L}[u(t-c)] = \begin{cases} \dfrac{e^{-cs}}{s} & \text{for} \quad c \geqslant 0, \\ \dfrac{1}{s} & \text{for} \quad c < 0. \end{cases}$$

Proof of Theorem 4.3.2: Consider the case $c \geqslant 0$. The Laplace transform is

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_c^\infty e^{-st}\, dt,$$

where we used that the step function vanishes for $t < c$. Now compute the improper integral,

$$\mathcal{L}[u(t-c)] = \lim_{N\to\infty} -\frac{1}{s}\big(e^{-Ns} - e^{-cs}\big) = \frac{e^{-cs}}{s} \quad \Rightarrow \quad \mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}.$$

Consider now the case of $c < 0$. The step function is identically equal to one in the domain of integration of the Laplace transform, which is $[0, \infty)$, hence

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_0^\infty e^{-st}\, dt = \mathcal{L}[1] = \frac{1}{s}.$$

This establishes the Theorem. □

Example 4.3.4. Compute $\mathcal{L}[3\,u(t-2)]$.

Solution: The Laplace transform is a linear operation, so

$$\mathcal{L}[3\,u(t-2)] = 3\,\mathcal{L}[u(t-2)],$$

and the Theorem 4.3.2 above implies that $\mathcal{L}[3\,u(t-2)] = \dfrac{3\,e^{-2s}}{s}$. ◁

Remarks:

(a) The LT is an invertible transformation in the set of functions we work in our class.

(b) $\mathcal{L}[f] = F \quad \Leftrightarrow \quad \mathcal{L}^{-1}[F] = f$.

Example 4.3.5. Compute $\mathcal{L}^{-1}\Big[\dfrac{e^{-3s}}{s}\Big]$.

Solution: Theorem 4.3.2 says that $\dfrac{e^{-3s}}{s} = \mathcal{L}[u(t-3)]$, so $\mathcal{L}^{-1}\Big[\dfrac{e^{-3s}}{s}\Big] = u(t-3)$. ◁

4.3.3. Translation Identities.

We now introduce two properties relating the Laplace transform and translations. The first property relates the Laplace transform of a translation with a multiplication by an exponential. The second property can be thought as the inverse of the first one.

Theorem 4.3.3 (Translation Identities). *If $\mathcal{L}[f(t)](s)$ exists for $s > a$, then*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)], \qquad s > a, \qquad c \geqslant 0 \tag{4.3.3}$$

$$\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)](s-c), \qquad s > a + c, \qquad c \in \mathbb{R}. \tag{4.3.4}$$

Example 4.3.6. Take $f(t) = \cos(t)$ and write the equations given the Theorem above.

Solution:

$$\mathcal{L}[\cos(t)] = \frac{s}{s^2+1} \quad \Rightarrow \quad \begin{cases} \mathcal{L}[u(t-c)\,\cos(t-c)] = e^{-cs}\,\dfrac{s}{s^2+1} \\[2ex] \mathcal{L}[e^{ct}\cos(t)] = \dfrac{(s-c)}{(s-c)^2+1}. \end{cases}$$

◁

Remarks:

(a) We can highlight the main idea in the theorem above as follows:

$$\mathcal{L}\big[\text{right-translation } (uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big),$$

$$\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big).$$

(b) Denoting $F(s) = \mathcal{L}[f(t)]$, then an equivalent expression for Eqs. (4.3.3)-(4.3.4) is

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,F(s),$$

$$\mathcal{L}[e^{ct}f(t)] = F(s-c).$$

(c) The inverse form of Eqs. (4.3.3)-(4.3.4) is given by,

$$\mathcal{L}^{-1}[e^{-cs}\,F(s)] = u(t-c)f(t-c), \tag{4.3.5}$$

$$\mathcal{L}^{-1}[F(s-c)] = e^{ct}f(t). \tag{4.3.6}$$

(d) Eq. (4.3.4) holds for all $c \in \mathbb{R}$, while Eq. (4.3.3) holds only for $c \geqslant 0$.
(e) Show that in the case that $c < 0$ the following equation holds,

$$\mathcal{L}[u(t+|c|)f(t+|c|)] = e^{|c|s}\Big(\mathcal{L}[f(t)] - \int_0^{|c|} e^{-st}\,f(t)\,dt\Big).$$

Proof of Theorem 4.3.3: The proof is again based in a change of the integration variable. We start with Eq. (4.3.3), as follows,

$$\begin{aligned}\mathcal{L}[u(t-c)f(t-c)] &= \int_0^\infty e^{-st}u(t-c)f(t-c)\,dt\\ &= \int_c^\infty e^{-st}f(t-c)\,dt, \qquad \tau = t-c, \qquad d\tau = dt, \qquad c \geqslant 0,\\ &= \int_0^\infty e^{-s(\tau+c)}f(\tau)\,d\tau\\ &= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau\\ &= e^{-cs}\,\mathcal{L}[f(t)], \qquad s > a.\end{aligned}$$

The proof of Eq. (4.3.4) is a bit simpler, since

$$\mathcal{L}\big[e^{ct}f(t)\big] = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = \mathcal{L}[f(t)](s-c),$$

which holds for $s - c > a$. This establishes the Theorem. □

Example 4.3.7. Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: Both $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$ and $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$ imply

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = \dfrac{a\,e^{-2s}}{s^2+a^2}$. ◁

Example 4.3.8. Compute $\mathcal{L}\big[e^{3t}\,\sin(at)\big]$.

Solution: Since $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$, then we get

$$\mathcal{L}\big[e^{3t}\,\sin(at)\big] = \frac{a}{(s-3)^2+a^2}, \qquad s > 3.$$

◁

Example 4.3.9. Compute both $\mathcal{L}\big[u(t-2)\,\cos\big(a(t-2)\big)\big]$ and $\mathcal{L}\big[e^{3t}\,\cos(at)\big]$.

Solution: Since $\mathcal{L}\big[\cos(at)\big] = \dfrac{s}{s^2+a^2}$, then

$$\mathcal{L}\big[u(t-2)\,\cos\big(a(t-2)\big)\big] = e^{-2s}\,\frac{s}{(s^2+a^2)}, \qquad \mathcal{L}\big[e^{3t}\,\cos(at)\big] = \frac{(s-3)}{(s-3)^2+a^2}.$$

◁

Example 4.3.10. Find the Laplace transform of the function

$$f(t) = \begin{cases} 0 & t < 1, \\ (t^2 - 2t + 2) & t \geqslant 1. \end{cases} \tag{4.3.7}$$

Solution: The idea is to rewrite function f so we can use the Laplace transform Table 1, in § 4.1 to compute its Laplace transform. Since the function f vanishes for all $t < 1$, we use step functions to write f as

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Now, notice that completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t-1)^2 + 1.$$

The polynomial is a parabola t^2 translated to the right and up by one. This is a discontinuous function, as it can be seen in Fig. 5.

So the function f can be written as follows,

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since we know that $\mathcal{L}[t^2] = \dfrac{2}{s^3}$, then Eq. (4.3.3) implies

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)\,(t-1)^2] + \mathcal{L}[u(t-1)]$$
$$= e^{-s}\,\frac{2}{s^3} + e^{-s}\,\frac{1}{s}$$

so we get

$$\mathcal{L}[f(t)] = \frac{e^{-s}}{s^3}\big(2 + s^2\big).$$

◁

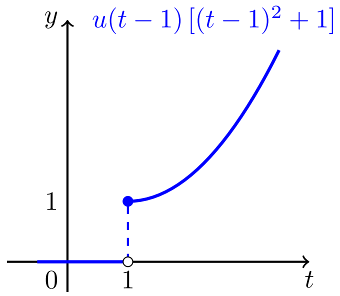


FIGURE 5. Function f given in Eq. (4.3.7).

Example 4.3.11. Find the function f such that $\mathcal{L}[f(t)] = \dfrac{e^{-4s}}{s^2+5}$.

Solution: Notice that

$$\mathcal{L}[f(t)] = e^{-4s}\left(\frac{1}{s^2+5}\right) \quad \Rightarrow \quad \mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,e^{-4s}\left(\frac{\sqrt{5}}{s^2+\big(\sqrt{5}\big)^2}\right).$$

Recall that $\mathcal{L}[\sin(at)] = \dfrac{a}{(s^2+a^2)}$, then

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,e^{-4s}\,\mathcal{L}[\sin(\sqrt{5}t)].$$

But the translation identity

$$e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)f(t-c)]$$

implies

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,\mathcal{L}\big[u(t-4)\sin\big(\sqrt{5}\,(t-4)\big)\big],$$

hence we obtain

$$f(t) = \frac{1}{\sqrt{5}}\,u(t-4)\sin\big(\sqrt{5}\,(t-4)\big).$$

◁

Example 4.3.12. Find the function $f(t)$ such that $\mathcal{L}[f(t)] = \dfrac{(s-1)}{(s-2)^2+3}$.

Solution: We first rewrite the right-hand side above as follows,

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{(s-1-1+1)}{(s-2)^2+3}\\
&= \frac{(s-2)}{(s-2)^2+3} + \frac{1}{(s-2)^2+3}\\
&= \frac{(s-2)}{(s-2)^2+\big(\sqrt{3}\big)^2} + \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s-2)^2+\big(\sqrt{3}\big)^2}\\
&= \mathcal{L}[\cos(\sqrt{3}\,t)](s-2) + \frac{1}{\sqrt{3}}\,\mathcal{L}[\sin(\sqrt{3}\,t)](s-2).
\end{aligned}$$

But the translation identity $\mathcal{L}[f(t)](s-c) = \mathcal{L}[e^{ct}f(t)]$ implies

$$\mathcal{L}[f(t)] = \mathcal{L}\big[e^{2t}\cos\big(\sqrt{3}\,t\big)\big] + \frac{1}{\sqrt{3}}\mathcal{L}\big[e^{2t}\sin\big(\sqrt{3}\,t\big)\big].$$

So, we conclude that

$$f(t) = \frac{e^{2t}}{\sqrt{3}}\Big[\sqrt{3}\,\cos\big(\sqrt{3}\,t\big) + \sin\big(\sqrt{3}\,t\big)\Big].$$

◁

Example 4.3.13. Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

Solution: Since $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2-a^2}\Big] = \sinh(at)$ and $\mathcal{L}^{-1}\big[e^{-cs}\,\hat{f}(s)\big] = u(t-c)\,f(t-c)$, then

$$\mathcal{L}^{-1}\Big[\frac{2e^{-3s}}{s^2-4}\Big] = \mathcal{L}^{-1}\Big[e^{-3s}\,\frac{2}{s^2-4}\Big] \quad\Rightarrow\quad \mathcal{L}^{-1}\Big[\frac{2e^{-3s}}{s^2-4}\Big] = u(t-3)\,\sinh\big(2(t-3)\big).$$

◁

Example 4.3.14. Find a function f such that $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{s^2+s-2}$.

Solution: Since the right hand side above does not appear in the Laplace transform Table in § 4.1, we need to simplify it in an appropriate way. The plan is to rewrite the denominator of the rational function $1/(s^2+s-2)$, so we can use partial fractions to simplify this rational function. We first find out whether this denominator has real or complex roots:

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad\Rightarrow\quad \begin{cases} s_+ = 1,\\ s_- = -2.\end{cases}$$

We are in the case of real roots, so we rewrite

$$s^2+s-2 = (s-1)\,(s+2).$$

The partial fraction decomposition in this case is given by

$$\frac{1}{(s-1)\,(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)} = \frac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)} \quad \Rightarrow \quad \begin{cases} a+b=0, \\ 2a-b=1. \end{cases}$$

The solution is $a = 1/3$ and $b = -1/3$, so we arrive to the expression

$$\mathcal{L}[f(t)] = \frac{1}{3}\,e^{-2s}\,\frac{1}{s-1} - \frac{1}{3}\,e^{-2s}\,\frac{1}{s+2}.$$

Recalling that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a},$$

and Eq. (4.3.3) we obtain the equation

$$\mathcal{L}[f(t)] = \frac{1}{3}\,\mathcal{L}\big[u(t-2)\,e^{(t-2)}\big] - \frac{1}{3}\,\mathcal{L}\big[u(t-2)\,e^{-2(t-2)}\big]$$

which leads to the conclusion:

$$f(t) = \frac{1}{3}\,u(t-2)\Big[e^{(t-2)} - e^{-2(t-2)}\Big].$$

◁

4.3.4. Solving Differential Equations. The last three examples in this section show how to use the methods presented above to solve differential equations with discontinuous source functions.

Example 4.3.15. Use the Laplace transform to find the solution of the initial value problem

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: We compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

We introduce the initial condition $y(0) = 3$ into equation above,

$$\mathcal{L}[y] = \frac{3}{(s+2)} + e^{-4s}\,\frac{1}{s(s+2)} \quad \Rightarrow \quad \mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\frac{1}{s(s+2)}.$$

We need to invert the Laplace transform on the last term on the right hand side in equation above. We use the partial fraction decomposition on the rational function above, as follows

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)} \quad \Rightarrow \quad \begin{cases} a+b=0, \\ \quad 2a=1. \end{cases}$$

We conclude that $a = 1/2$ and $b = -1/2$, so

$$\frac{1}{s(s+2)} = \frac{1}{2}\Big[\frac{1}{s} - \frac{1}{(s+2)}\Big].$$

We then obtain

$$\begin{aligned} \mathcal{L}[y] &= 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\Big] \\ &= 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\big[u(t-4)\,e^{-2(t-4)}\big]\Big). \end{aligned}$$

Hence, we conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2}\,u(t-4)\Big[1 - e^{-2(t-4)}\Big].$$

$\lhd$

Example 4.3.16. Use the Laplace transform to find the solution to the initial value problem

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad b(t) = \begin{cases} 1 & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4.3.8}$$

Solution: From Fig. 6, the source function b can be written as

$$b(t) = u(t) - u(t-\pi).$$

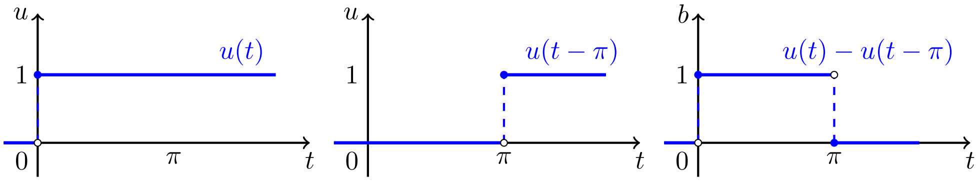


FIGURE 6. The graph of the u, its translation and b as given in Eq. (4.3.8).

The last expression for b is particularly useful to find its Laplace transform,

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)] = \frac{1}{s} + e^{-\pi s}\,\frac{1}{s} \quad\Rightarrow\quad \mathcal{L}[b(t)] = (1 - e^{-\pi s})\,\frac{1}{s}.$$

Now Laplace transform the whole equation,

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[b].$$

Since the initial condition are $y(0) = 0$ and $y'(0) = 0$, we obtain

$$\Big(s^2 + s + \frac{5}{4}\Big)\,\mathcal{L}[y] = \big(1 - e^{-\pi s}\big)\,\frac{1}{s} \quad\Rightarrow\quad \mathcal{L}[y] = \big(1 - e^{-\pi s}\big)\,\frac{1}{s\,\Big(s^2 + s + \frac{5}{4}\Big)}.$$

Introduce the function

$$H(s) = \frac{1}{s\,\Big(s^2 + s + \frac{5}{4}\Big)} \quad\Rightarrow\quad y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s}\,H(s)].$$

That is, we only need to find the inverse Laplace transform of H. We use partial fractions to simplify the expression of H. We first find out whether the denominator has real or complex roots:

$$s^2 + s + \frac{5}{4} = 0 \quad\Rightarrow\quad s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big],$$

so the roots are complex valued. An appropriate partial fraction decomposition is

$$H(s) = \frac{1}{s\,\big(s^2 + s + \frac{5}{4}\big)} = \frac{a}{s} + \frac{(bs + c)}{\big(s^2 + s + \frac{5}{4}\big)}$$

Therefore, we get

$$1 = a\,\Big(s^2 + s + \frac{5}{4}\Big) + s\,(bs + c) = (a + b)\,s^2 + (a + c)\,s + \frac{5}{4}a.$$

This equation implies that a, b, and c, satisfy the equations

$$a + b = 0, \qquad a + c = 0, \qquad \frac{5}{4}\,a = 1.$$

The solution is, $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$. Hence, we have found that,

$$H(s) = \frac{1}{\big(s^2 + s + \frac{5}{4}\big)\,s} = \frac{4}{5}\Big[\frac{1}{s} - \frac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big]$$

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

Replace this expression in the definition of H, that is,

$$H(s) = \frac{4}{5}\Big[\frac{1}{s} - \frac{(s+1)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big]$$

Rewrite the polynomial in the numerator,

$$(s+1) = \Big(s + \frac{1}{2} + \frac{1}{2}\Big) = \Big(s + \frac{1}{2}\Big) + \frac{1}{2},$$

hence we get

$$H(s) = \frac{4}{5}\Big[\frac{1}{s} - \frac{\big(s + \frac{1}{2}\big)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]} - \frac{1}{2}\,\frac{1}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$$

Use the Laplace transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{5}\Big[\mathcal{L}[1] - \mathcal{L}\big[e^{-t/2}\cos(t)\big] - \frac{1}{2}\mathcal{L}[e^{-t/2}\sin(t)]\Big],$$

equivalently

$$H(s) = \mathcal{L}\Big[\frac{4}{5}\Big(1 - e^{-t/2}\cos(t) - \frac{1}{2}e^{-t/2}\sin(t)\Big)\Big].$$

Denote

$$h(t) = \frac{4}{5}\Big[1 - e^{-t/2}\cos(t) - \frac{1}{2}e^{-t/2}\sin(t)\Big]. \quad \Rightarrow \quad H(s) = \mathcal{L}[h(t)].$$

Recalling $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$, we obtain $\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)]$, that is,

$$y(t) = h(t) + u(t - \pi)h(t - \pi).$$

◁

Example 4.3.17. Use the Laplace transform to find the solution to the initial value problem

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad g(t) = \begin{cases} \sin(t) & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4.3.9}$$

Solution: From Fig. 7, the source function g can be written as the following product,

$$g(t) = \big[u(t) - u(t - \pi)\big]\,\sin(t),$$

since $u(t) - u(t - \pi)$ is a box function, taking value one in the interval $[0, \pi]$ and zero on the complement. Finally, notice that the equation $\sin(t) = -\sin(t - \pi)$ implies that the function g can be expressed as follows,

$$g(t) = u(t)\,\sin(t) - u(t - \pi)\,\sin(t) \quad \Rightarrow \quad g(t) = u(t)\,\sin(t) + u(t - \pi)\,\sin(t - \pi).$$

The last expression for g is particularly useful to find its Laplace transform,

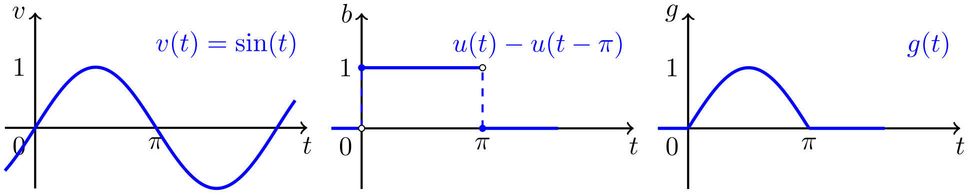


FIGURE 7. The graph of the sine function, a square function $u(t)-u(t-\pi)$ and the source function g given in Eq. (4.3.9).

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

With this last transform is not difficult to solve the differential equation. As usual, Laplace transform the whole equation,

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

Since the initial condition are $y(0) = 0$ and $y'(0) = 0$, we obtain

$$\Big(s^2 + s + \frac{5}{4}\Big)\,\mathcal{L}[y] = \big(1 + e^{-\pi s}\big)\,\frac{1}{(s^2+1)} \quad \Rightarrow \quad \mathcal{L}[y] = \big(1 + e^{-\pi s}\big)\,\frac{1}{\big(s^2 + s + \frac{5}{4}\big)\,(s^2+1)}.$$

Introduce the function

$$H(s) = \frac{1}{\big(s^2 + s + \frac{5}{4}\big)\,(s^2+1)} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)].$$

That is, we only need to find the Inverse Laplace transform of H. We use partial fractions to simplify the expression of H. We first find out whether the denominator has real or complex roots:

$$s^2 + s + \frac{5}{4} = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big],$$

so the roots are complex valued. An appropriate partial fraction decomposition is

$$H(s) = \frac{1}{\big(s^2 + s + \frac{5}{4}\big)\,(s^2+1)} = \frac{(as+b)}{\big(s^2 + s + \frac{5}{4}\big)} + \frac{(cs+d)}{(s^2+1)}.$$

Therefore, we get

$$1 = (as+b)(s^2+1) + (cs+d)\Big(s^2 + s + \frac{5}{4}\Big),$$

equivalently,

$$1 = (a+c)\,s^3 + (b+c+d)\,s^2 + \Big(a + \frac{5}{4}\,c + d\Big)\,s + \Big(b + \frac{5}{4}\,d\Big).$$

This equation implies that a, b, c, and d, are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}\,c + d = 0, \quad b + \frac{5}{4}\,d = 1.$$

Here is the solution to this system:

$$a = \frac{16}{17}, \qquad b = \frac{12}{17}, \qquad c = -\frac{16}{17}, \qquad d = \frac{4}{17}.$$

We have found that,

$$H(s) = \frac{4}{17}\,\Big[\frac{(4s+3)}{\big(s^2 + s + \frac{5}{4}\big)} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\, s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

$$H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3 = 4\Big(s + \frac{1}{2}\Big) + 1,$$

hence we get

$$H(s) = \frac{4}{17}\Big[4\,\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} + \frac{1}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\Big].$$

Use the Laplace transform Table in 1 to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\Big[4\,\mathcal{L}\big[e^{-t/2}\cos(t)\big] + \mathcal{L}\big[e^{-t/2}\sin(t)\big] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\Big],$$

equivalently

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big] \quad\Rightarrow\quad H(s) = \mathcal{L}[h(t)].$$

Recalling $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$, we obtain $\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)]$, that is,

$$y(t) = h(t) + u(t-\pi)h(t-\pi).$$

$\lhd$

4.3.5. Exercises.

4.3.1.- .

4.3.2.- .

4.4. Generalized Sources

We introduce a generalized function—the Dirac delta. We define the Dirac delta as a limit $n \to \infty$ of a particular sequence of functions, $\{\delta_n\}$. We will see that this limit is a function on the domain $\mathbb{R} - \{0\}$, but it is not a function on $\mathbb{R}$. For that reason we call this limit a generalized function—the Dirac delta generalized function.

We will show that each element in the sequence $\{\delta_n\}$ has a Laplace transform, and this sequence of Laplace transforms $\{\mathcal{L}[\delta_n]\}$ has a limit as $n \to \infty$. We use this limit of Laplace transforms to define the Laplace transform of the Dirac delta.

We will solve differential equations having the Dirac delta generalized function as source. Such differential equations appear often when one describes physical systems with impulsive forces—forces acting on a very short time but transfering a finite momentum to the system. Dirac's delta is tailored to model impulsive forces.

4.4.1. Sequence of Functions and the Dirac Delta.

A sequence of functions is a sequence whose elements are functions. If each element in the sequence is a continuous function, we say that this is a sequence of continuous functions. Given a sequence of functions $\{y_n\}$, we compute the $\lim_{n\to\infty} y_n(t)$ for a fixed t. The limit depends on t, so it is a function of t, and we write it as

$$\lim_{n\to\infty} y_n(t) = y(t).$$

The domain of the limit function y is smaller or equal to the domain of the y_n. The limit of a sequence of continuous functions may or may not be a continuous function.

Example 4.4.1. The limit of the sequence below is a continuous function,

$$\Big\{ f_n(t) = \sin\Big(\Big(1 + \frac{1}{n}\Big)t\Big) \Big\} \to \sin(t) \quad \text{as} \quad n \to \infty.$$

As usual in this section, the limit is computed for each fixed value of t. ◁

However, not every sequence of continuous functions has a continuous function as a limit.

Example 4.4.2. Consider now the following sequence, $\{u_n\}$, for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases} \tag{4.4.1}$$

This is a sequence of continuous functions whose limit is a discontinuous function. From the few graphs in Fig. 8 we can see that the limit $n \to \infty$ of the sequence above is a step function, indeed, $\lim_{n\to\infty} u_n(t) = \tilde{u}(t)$, where

$$\tilde{u}(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ 1 & \text{for} \quad t > 0. \end{cases}$$

We used a tilde in the name $\tilde{u}$ because this step function is not the same we defined in the previous section. The step u in § 4.3 satisfied $u(0) = 1$. ◁

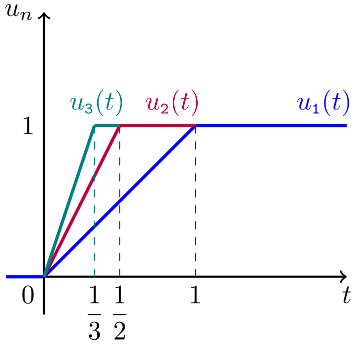


Figure 8. A few functions in the sequence $\{u_n\}$.

Exercise: Find a sequence $\{u_n\}$ so that its limit is the step function u defined in § 4.3.

Although every function in the sequence $\{u_n\}$ is continuous, the limit $\tilde{u}$ is a discontinuous function. It is not difficult to see that one can construct sequences of continuous functions having no limit at all. A similar situation happens when one considers sequences of piecewise discontinuous functions. In this case the limit could be a continuous function, a piecewise discontinuous function, or not a function at all.

We now introduce a particular sequence of piecewise discontinuous functions with domain $\mathbb{R}$ such that the limit as $n \to \infty$ does not exist for all values of the independent variable t. The limit of the sequence is not a function with domain $\mathbb{R}$. In this case, the limit is a new type of object that we will call Dirac's delta generalized function. Dirac's delta is the limit of a sequence of particular bump functions.

Definition 4.4.1. *The* ***Dirac delta*** *generalized function is the limit*

$$\delta(t) = \lim_{n\to\infty} \delta_n(t),$$

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

$$\delta_n(t) = n\Big[u(t) - u\big(t - \frac{1}{n}\big)\Big]. \tag{4.4.2}$$

The sequence of bump functions introduced above can be rewritten as follows,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t < \dfrac{1}{n} \\ 0, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} 0 & \text{for} \quad t \neq 0, \\ \infty & \text{for} \quad t = 0. \end{cases}$$

Remark: It can be shown that there exist infinitely many sequences $\{\tilde{\delta}_n\}$ such that their limit as $n \to \infty$ is Dirac's delta. For example, another sequence is

$$\begin{aligned} \tilde{\delta}_n(t) &= n\Big[u\big(t + \frac{1}{2n}\big) - u\big(t - \frac{1}{2n}\big)\Big] \\ &= \begin{cases} 0, & t < -\dfrac{1}{2n} \\ n, & -\dfrac{1}{2n} \leqslant t \leqslant \dfrac{1}{2n} \\ 0, & t > \dfrac{1}{2n}. \end{cases} \end{aligned}$$

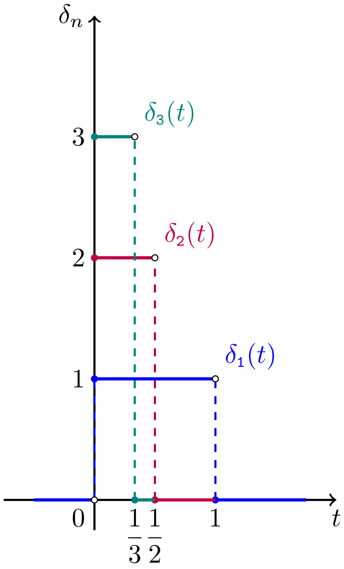


FIGURE 9. A few functions in the sequence $\{\delta_n\}$.

The Dirac delta generalized function is the function identically zero on the domain $\mathbb{R} - \{0\}$. Dirac's delta is not defined at $t = 0$, since the limit diverges at that point. If we shift each element in the sequence by a real number c, then we define

$$\delta(t - c) = \lim_{n\to\infty} \delta_n(t - c), \qquad c \in \mathbb{R}.$$

This shifted Dirac's delta is identically zero on $\mathbb{R} - \{c\}$ and diverges at $t = c$. If we shift the graphs given in Fig. 9 by any real number c, one can see that

$$\int_c^{c+1} \delta_n(t - c)\, dt = 1$$

for every $n \geqslant 1$. Therefore, the sequence of integrals is the constant sequence, $\{1, 1, \cdots\}$, which has a trivial limit, 1, as $n \to \infty$. This says that the divergence at $t = c$ of the sequence $\{\delta_n\}$ is of a very particular type. The area below the graph of the sequence elements is always the same. We can say that this property of the sequence provides the main defining property of the Dirac delta generalized function.

Using a limit procedure one can generalize several operations from a sequence to its limit. For example, translations, linear combinations, and multiplications of a function by a generalized function, integration and Laplace transforms.

Definition 4.4.2. *We introduce the following operations on the Dirac delta:*

$$\begin{gathered} f(t)\, \delta(t-c) + g(t)\, \delta(t-c) = \lim_{n\to\infty} \big[f(t)\, \delta_n(t-c) + g(t)\, \delta_n(t-c)\big], \\ \int_a^b \delta(t-c)\, dt = \lim_{n\to\infty} \int_a^b \delta_n(t-c)\, dt, \\ \mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]. \end{gathered}$$

Remark: The notation in the definitions above could be misleading. In the left hand sides above we use the same notation as we use on functions, although Dirac's delta is not a function on $\mathbb{R}$. Take the integral, for example. When we integrate a function f, the integration symbol means "take a limit of Riemann sums", that is,

$$\int_a^b f(t)\, dt = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\, \Delta x, \qquad x_i = a + i\, \Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

However, when f is a generalized function in the sense of a limit of a sequence of functions $\{f_n\}$, then by the integration symbol we mean to compute a different limit,

$$\int_a^b f(t)\, dt = \lim_{n\to\infty} \int_a^b f_n(t)\, dt.$$

We use the same symbol, the integration, to mean two different things, depending whether we integrate a function or a generalized function. This remark also holds for all the operations we introduce on generalized functions, specially the Laplace transform, that will be often used in the rest of this section.

4.4.2. Computations with the Dirac Delta.

Once we have the definitions of operations involving the Dirac delta, we can actually compute these limits. The following statement summarizes few interesting results. The first formula below says that the infinity we found in the definition of Dirac's delta is of a very particular type; that infinity is such that Dirac's delta is integrable, in the sense defined above, with integral equal one.

Theorem 4.4.3. *For every $c \in \mathbb{R}$ and $\epsilon > 0$ holds,* $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\, dt = 1.$

Proof of Theorem 4.4.3: The integral of a Dirac's delta generalized function is computed as a limit of integrals,

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = \lim_{n\to\infty} \int_{c-\epsilon}^{c+\epsilon} \delta_n(t-c)\,dt.$$

If we choose $n > 1/\epsilon$, equivalently $1/n < \epsilon$, then the domain of the functions in the sequence is inside the interval $(c-\epsilon, c+\epsilon)$, and we can write

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = \lim_{n\to\infty} \int_{c}^{c+\frac{1}{n}} n\,dt, \qquad \text{for} \qquad \frac{1}{n} < \epsilon.$$

Then it is simple to compute

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = \lim_{n\to\infty} n\Big(c + \frac{1}{n} - c\Big) = \lim_{n\to\infty} 1 = 1.$$

This establishes the Theorem. □

The next result is also deeply related with the defining property of the Dirac delta—the sequence functions have all graphs of unit area.

Theorem 4.4.4. *If f is continuous on (a,b) and $c \in (a,b)$, then* $\displaystyle\int_a^b f(t)\,\delta(t-c)\,dt = f(c)$.

Proof of Theorem 4.4.4: We again compute the integral of a Dirac's delta as a limit of a sequence of integrals,

$$\begin{aligned}
\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} \int_a^b \delta_n(t-c)\,f(t)\,dt \\
&= \lim_{n\to\infty} \int_a^b n\Big[u(t-c) - u\Big(t-c-\frac{1}{n}\Big)\Big]\,f(t)\,dt \\
&= \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\,f(t)\,dt, \qquad \frac{1}{n} < (b-c),
\end{aligned}$$

To get the last line we used that $c \in [a,b]$. Let F be any primitive of f, so $F(t) = \int f(t)\,dt$. Then we can write,

$$\begin{aligned}
\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} n\Big[F\Big(c+\frac{1}{n}\Big) - F(c)\Big] \\
&= \lim_{n\to\infty} \frac{1}{\big(\frac{1}{n}\big)}\Big[F\Big(c+\frac{1}{n}\Big) - F(c)\Big] \\
&= F'(c) \\
&= f(c).
\end{aligned}$$

This establishes the Theorem. □

In our next result we compute the Laplace transform of the Dirac delta. We give two proofs of this result. In the first proof we use the previous theorem. In the second proof we use the same idea used to prove the previous theorem.

Theorem 4.4.5. *For all $s \in \mathbb{R}$ holds* $\mathcal{L}[\delta(t-c)] = \begin{cases} e^{-cs} & \text{for} \quad c \geqslant 0, \\ 0 & \text{for} \quad c < 0. \end{cases}$

First Proof of Theorem 4.4.5: We use the previous theorem on the integral that defines a Laplace transform. Although the previous theorem applies to definite integrals, not to improper integrals, it can be extended to cover improper integrals. In this case we get

$$\mathcal{L}[\delta(t-c)] = \int_0^\infty e^{-st}\,\delta(t-c)\,dt = \begin{cases} e^{-cs} & \text{for} \quad c \geqslant 0,\\ 0 & \text{for} \quad c < 0, \end{cases}$$

This establishes the Theorem. □

Second Proof of Theorem 4.4.5: The Laplace transform of a Dirac's delta is computed as a limit of Laplace transforms,

$$\begin{aligned} \mathcal{L}[\delta(t-c)] &= \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] \\ &= \lim_{n\to\infty} \mathcal{L}\Big[n\big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\big]\Big] \\ &= \lim_{n\to\infty} \int_0^\infty n\big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\big]\,e^{-st}\,dt. \end{aligned}$$

The case $c < 0$ is simple. For $\dfrac{1}{n} < |c|$ holds

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_0^\infty 0\,dt \quad \Rightarrow \quad \mathcal{L}[\delta(t-c)] = 0, \qquad \text{for } s \in \mathbb{R}, \quad c < 0.$$

Consider now the case $c \geqslant 0$. We then have,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,e^{-st}\,dt.$$

For $s = 0$ we get

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,dt = 1 \quad \Rightarrow \quad \mathcal{L}[\delta(t-c)] = 1 \quad \text{for } s = 0, \quad c \geqslant 0.$$

In the case that $s \neq 0$ we get,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,e^{-st}\,dt = \lim_{n\to\infty} -\frac{n}{s}\big(e^{-cs} - e^{-(c+\frac{1}{n})s}\big) = e^{-cs}\lim_{n\to\infty}\frac{(1-e^{-\frac{s}{n}})}{\left(\dfrac{s}{n}\right)}.$$

The limit on the last line above is a singular limit of the form $\frac{0}{0}$, so we can use the l'Hôpital rule to compute it, that is,

$$\lim_{n\to\infty}\frac{(1-e^{-\frac{s}{n}})}{\left(\dfrac{s}{n}\right)} = \lim_{n\to\infty}\frac{\left(-\dfrac{s}{n^2}\,e^{-\frac{s}{n}}\right)}{\left(-\dfrac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We then obtain,

$$\mathcal{L}[\delta(t-c)] = e^{-cs} \quad \text{for } s \neq 0, \quad c \geqslant 0.$$

This establishes the Theorem. □

4.4.3. Applications of the Dirac Delta.

Dirac's delta generalized functions describe *impulsive forces* in mechanical systems, such as the force done by a stick hitting a marble. An impulsive force acts on an infinitely short time and transmits a finite momentum to the system.

Example 4.4.3. Use Newton's equation of motion and Dirac's delta to describe the change of momentum when a particle is hit by a hammer.

Solution: A point particle with mass m, moving on one space direction, x, with a force F acting on it is described by

$$ma = F \quad \Leftrightarrow \quad mx''(t) = F(t, x(t)),$$

where $x(t)$ is the particle position as function of time, $a(t) = x''(t)$ is the particle acceleration, and we will denote $v(t) = x'(t)$ the particle velocity. We saw in § 1.1 that Newton's second law of motion is a second order differential equation for the position function x. Now it is more convenient to use the *particle momentum*, $p = mv$, to write the Newton's equation,

$$mx'' = mv' = (mv)' = F \quad \Rightarrow \quad p' = F.$$

So the force F changes the momentum, P. If we integrate on an interval $[t_1, t_2]$ we get

$$\Delta p = p(t_2) - p(t_1) = \int_{t_1}^{t_2} F(t, x(t))\, dt.$$

Suppose that an impulsive force is acting on a particle at t_0 transmitting a finite momentum, say p_0. This is where the Dirac delta is uselful for, because we can write the force as

$$F(t) = p_0\, \delta(t - t_0),$$

then $F = 0$ on $\mathbb{R} - \{t_0\}$ and the momentum transferred to the particle by the force is

$$\Delta p = \int_{t_0 - \Delta t}^{t_0 + \Delta t} p_0\, \delta(t - t_0)\, dt = p_0.$$

The momentum tranferred is $\Delta p = p_0$, but the force is identically zero on $\mathbb{R} - \{t_0\}$. We have transferred a finite momentum to the particle by an interaction at a single time t_0. ◁

4.4.4. The Impulse Response Function. We now want to solve differential equations with the Dirac delta as a source. But there is a particular type of solutions that will be important later on—solutions to initial value problems with the Dirac delta source and zero initial conditions. We give these solutions a particular name.

Definition 4.4.6. *The* ***impulse response function*** *at the point $c \geqslant 0$ of the constant coefficients linear operator $L(y) = y'' + a_1\, y' + a_0\, y$, is the solution y_δ of*

$$L(y_\delta) = \delta(t - c), \qquad y_\delta(0) = 0, \qquad y'_\delta(0) = 0.$$

Remark: Impulse response functions are also called *fundamental solutions.*

Theorem 4.4.7. *The function y_δ is the impulse response function at $c \geqslant 0$ of the constant coefficients operator $L(y) = y'' + a_1\, y' + a_0\, y$ iff holds*

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

where p is the characteristic polynomial of L.

Remark: The impulse response function y_δ at $c = 0$ satifies

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Proof of Theorem 4.4.7: Compute the Laplace transform of the differential equation for for the impulse response function y_δ,

$$\mathcal{L}[y''] + a_1\,\mathcal{L}[y'] + a_0\,\mathcal{L}[y] = \mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Since the initial data for y_δ is trivial, we get

$$(s^2 + a_1 s + a_0)\,\mathcal{L}[y] = e^{-cs}.$$

Since $p(s) = s^2 + a_1 s + a_0$ is the characteristic polynomial of L, we get

$$\mathcal{L}[y] = \frac{e^{-cs}}{p(s)} \quad \Leftrightarrow \quad y(t) = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

All the steps in this calculation are if and only ifs. This establishes the Theorem. □

Example 4.4.4. Find the impulse response function at $t = 0$ of the linear operator

$$L(y) = y'' + 2y' + 2y.$$

Solution: We need to find the solution y_δ of the initial value problem

$$y_\delta'' + 2y_\delta' + 2y_\delta = \delta(t), \qquad y_\delta(0) = 0, \qquad y_\delta'(0) = 0.$$

Since the souce is a Dirac delta, we have to use the Laplace transform to solve this problem. So we compute the Laplace transform on both sides of the differential equation,

$$\mathcal{L}[y_\delta''] + 2\,\mathcal{L}[y_\delta'] + 2\,\mathcal{L}[y_\delta] = \mathcal{L}[\delta(t)] = 1 \quad \Rightarrow \quad (s^2 + 2s + 2)\,\mathcal{L}[y_\delta] = 1,$$

where we have introduced the initial conditions on the last equation above. So we obtain

$$\mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)}.$$

The denominator in the equation above has complex valued roots, since

$$s_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4-8}\big],$$

therefore, we complete squares $s^2 + 2s + 2 = (s+1)^2 + 1$. We need to solve the equation

$$\mathcal{L}[y_\delta] = \frac{1}{\big[(s+1)^2 + 1\big]} = \mathcal{L}[e^{-t}\sin(t)] \quad \Rightarrow \quad y_\delta(t) = e^{-t}\sin(t).$$

◁

Example 4.4.5. Find the impulse response function at $t = c \geqslant 0$ of the linear operator

$$L(y) = y'' + 2\,y' + 2\,y.$$

Solution: We need to find the solution y_δ of the initial value problem

$$y_\delta'' + 2\,y_\delta' + 2\,y_\delta = \delta(t-c), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

We have to use the Laplace transform to solve this problem because the source is a Dirac's delta generalized function. So, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y_\delta''] + 2\,\mathcal{L}[y_\delta'] + 2\,\mathcal{L}[y_\delta] = \mathcal{L}[\delta(t-c)].$$

Since the initial conditions are all zero and $c \geqslant 0$, we get

$$(s^2 + 2s + 2)\,\mathcal{L}[y_\delta] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right]$$

The denominator has complex roots. Then, it is convenient to complete the square in the denominator,

$$s^2 + 2s + 2 = \Big[s^2 + 2\Big(\frac{2}{2}\Big)\, s + 1\Big] - 1 + 2 = (s+1)^2 + 1.$$

Therefore, we obtain the expression,

$$\mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s+1)^2 + 1}.$$

Recall that $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2+1}$, and $\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct}\, f(t)]$. Then,

$$\frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\, \sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs}\, \mathcal{L}[e^{-t}\, \sin(t)].$$

Since for $c \geqslant 0$ holds $e^{-cs}\, \mathcal{L}[f](s) = \mathcal{L}[u(t-c)\, f(t-c)]$, we conclude that

$$y_\delta(t) = u(t-c)\, e^{-(t-c)}\, \sin(t-c).$$

$\lhd$

Example 4.4.6. Find the solution y to the initial value problem

$$y'' - y = -20\, \delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: The source is a generalized function, so we need to solve this problem using the Lapace transform. So we compute the Laplace transform of the differential equation,

$$\mathcal{L}[y''] - \mathcal{L}[y] = -20\, \mathcal{L}[\delta(t-3)] \quad \Rightarrow \quad (s^2-1)\, \mathcal{L}[y] - s = -20\, e^{-3s},$$

where in the second equation we have already introduced the initial conditions. We arrive to the equation

$$\mathcal{L}[y] = \frac{s}{(s^2-1)} - 20\, e^{-3s}\, \frac{1}{(s^2-1)} = \mathcal{L}[\cosh(t)] - 20\, \mathcal{L}[u(t-3)\, \sinh(t-3)],$$

which leads to the solution

$$y(t) = \cosh(t) - 20\, u(t-3)\, \sinh(t-3).$$

$\lhd$

Example 4.4.7. Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: We again Laplace transform both sides of the differential equation,

$$\mathcal{L}[y''] + 4\, \mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)] \quad \Rightarrow \quad (s^2+4)\, \mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s},$$

where in the second equation above we have introduced the initial conditions. Then,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)} \\
&= \frac{e^{-\pi s}}{2}\, \frac{2}{(s^2+4)} - \frac{e^{-2\pi s}}{2}\, \frac{2}{(s^2+4)} \\
&= \frac{1}{2}\mathcal{L}\Big[u(t-\pi)\, \sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\mathcal{L}\Big[u(t-2\pi)\, \sin\big[2(t-2\pi)\big]\Big].
\end{aligned}$$

The last equation can be rewritten as follows,

$$y(t) = \frac{1}{2}\, u(t-\pi)\, \sin\big[2(t-\pi)\big] - \frac{1}{2}\, u(t-2\pi)\, \sin\big[2(t-2\pi)\big],$$

which leads to the conclusion that

$$y(t) = \frac{1}{2}\,\big[u(t-\pi) - u(t-2\pi)\big]\,\sin(2t).$$

$\triangleleft$

4.4.5. Comments on Generalized Sources. We have used the Laplace transform to solve differential equations with the Dirac delta as a source function. It may be convenient to understand a bit more clearly what we have done, since the Dirac delta is not an ordinary function but a generalized function defined by a limit. Consider the following example.

Example 4.4.8. Find the impulse response function at $t = c > 0$ of the linear operator

$$L(y) = y'.$$

Solution: We need to solve the initial value problem

$$y'(t) = \delta(t-c), \qquad y(0) = 0.$$

In other words, we need to find a primitive of the Dirac delta. However, Dirac's delta is not even a function. Anyway, let us compute the Laplace transform of the equation, as we did in the previous examples,

$$\mathcal{L}[y'(t)] = \mathcal{L}[\delta(t-c)] \quad\Rightarrow\quad s\,\mathcal{L}[y(t)] - y(0) = e^{-cs} \quad\Rightarrow\quad \mathcal{L}[y(t)] = \frac{e^{-cs}}{s}.$$

But we know that

$$\frac{e^{-cs}}{s} = \mathcal{L}[u(t-c)] \quad\Rightarrow\quad \mathcal{L}[y(t)] = \mathcal{L}[u(t-c)] \quad\Rightarrow\quad y(t) = u(t-c).$$

$\triangleleft$

Looking at the differential equation $y'(t) = \delta(t-c)$ and at the solution $y(t) = u(t-c)$ one could like to write them together as

$$u'(t-c) = \delta(t-c). \tag{4.4.3}$$

But this is not correct, because the step function is a discontinuous function at $t = c$, hence not differentiable. What we have done is something different. We have found a sequence of functions u_n with the properties,

$$\lim_{n\to\infty} u_n(t-c) = u(t-c), \qquad \lim_{n\to\infty} u_n'(t-c) = \delta(t-c),$$

and we have called $y(t) = u(t-c)$. This is what we actually do when we solve a differential equation with a source defined as a limit of a sequence of functions, such as the Dirac delta. The Laplace transform method used on differential equations with generalized sources allows us to solve these equations without the need to write any sequence, which are hidden in the definitions of the Laplace transform of generalized functions. Let us solve the problem in the Example 4.4.8 one more time, but this time let us show where all the sequences actually are.

Example 4.4.9. Find the solution to the initial value problem

$$y'(t) = \delta(t-c), \qquad y(0) = 0, \qquad c > 0, \tag{4.4.4}$$

Solution: Recall that the Dirac delta is defined as a limit of a sequence of bump functions,

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c), \qquad \delta_n(t-c) = n\Big[u(t-c) - u\Big(t-c-\frac{1}{n}\Big)\Big], \qquad n = 1, 2, \cdots.$$

The problem we are actually solving involves a sequence and a limit,

$$y'(t) = \lim_{n\to\infty} \delta_n(t-c), \qquad y(0) = 0.$$

We start computing the Laplace transform of the differential equation,

$$\mathcal{L}[y'(t)] = \mathcal{L}[\lim_{n\to\infty} \delta_n(t-c)].$$

We have defined the Laplace transform of the limit as the limit of the Laplace transforms,

$$\mathcal{L}[y'(t)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)].$$

If the solution is at least piecewise differentiable, we can use the property

$$\mathcal{L}[y'(t)] = s\,\mathcal{L}[y(t)] - y(0).$$

Assuming that property, and the initial condition $y(0) = 0$, we get

$$\mathcal{L}[y(t)] = \frac{1}{s} \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] \quad \Rightarrow \quad \mathcal{L}[y(t)] = \lim_{n\to\infty} \frac{\mathcal{L}[\delta_n(t-c)]}{s}.$$

Introduce now the function $y_n(t) = u_n(t-c)$, given in Eq. (4.4.1), which for each n is the only continuous, piecewise differentiable, solution of the initial value problem

$$y_n'(t) = \delta_n(t-c), \qquad y_n(0) = 0.$$

It is not hard to see that this function u_n satisfies

$$\mathcal{L}[u_n(t)] = \frac{\mathcal{L}[\delta_n(t-c)]}{s}.$$

Therefore, using this formula back in the equation for y we get,

$$\mathcal{L}[y(t)] = \lim_{n\to\infty} \mathcal{L}[u_n(t)].$$

For continuous functions we can interchange the Laplace transform and the limit,

$$\mathcal{L}[y(t)] = \mathcal{L}[\lim_{n\to\infty} u_n(t)].$$

So we get the result,

$$y(t) = \lim_{n\to\infty} u_n(t) \quad \Rightarrow \quad y(t) = u(t-c).$$

We see above that we have found something more than just $y(t) = u(t-c)$. We have found

$$y(t) = \lim_{n\to\infty} u_n(t-c),$$

where the sequence elements u_n are continuous functions with $u_n(0) = 0$ and

$$\lim_{n\to\infty} u_n(t-c) = u(t-c), \qquad \lim_{n\to\infty} u_n'(t-c) = \delta(t-c),$$

Finally, derivatives and limits cannot be interchanged for u_n,

$$\lim_{n\to\infty} \big[u_n'(t-c)\big] \neq \big[\lim_{n\to\infty} u_n(t-c)\big]'$$

so it makes no sense to talk about y'. ◁

When the Dirac delta is defined by a sequence of functions, as we did in this section, the calculation needed to find impulse response functions must involve sequence of functions and limits. The Laplace transform method used on generalized functions allows us to hide all the sequences and limits. This is true not only for the derivative operator $L(y) = y'$ but for any second order differential operator with constant coefficients.

Definition 4.4.8. *A* ***solution*** *of the initial value problem with a Dirac's delta source*

$$y'' + a_1\, y' + a_0\, y = \delta(t-c), \qquad y(0) = y_0, \qquad y'(0) = y_1, \tag{4.4.5}$$

where a_1, a_0, y_0, y_1, and $c \in \mathbb{R}$, are given constants, is a function

$$y(t) = \lim_{n\to\infty} y_n(t),$$

where the functions y_n, with $n \geqslant 1$, are the unique solutions to the initial value problems

$$y_n'' + a_1\, y_n' + a_0\, y_n = \delta_n(t-c), \qquad y_n(0) = y_0, \qquad y_n'(0) = y_1, \tag{4.4.6}$$

and the source δ_n satisfy $\lim_{n\to\infty} \delta_n(t-c) = \delta(t-c)$.

The definition above makes clear what do we mean by a solution to an initial value problem having a generalized function as source, when the generalized function is defined as the limit of a sequence of functions. The following result says that the Laplace transform method used with generalized functions hides all the sequence computations.

Theorem 4.4.9. *The function y is solution of the initial value problem*

$$y'' + a_1\, y' + a_0\, y = \delta(t-c), \qquad y(0) = y_0, \qquad y'(0) = y_1, \qquad c \geqslant 0,$$

iff its Laplace transform satisfies the equation

$$\big(s^2\, \mathcal{L}[y] - sy_0 - y_1\big) + a_1\, \big(s\, \mathcal{L}[y] - y_0\big) - a_0\, \mathcal{L}[y] = e^{-cs}.$$

This Theorem tells us that to find the solution y to an initial value problem when the source is a Dirac's delta we have to apply the Laplace transform to the equation and perform the same calculations as if the Dirac delta were a function. This is the calculation we did when we computed the impulse response functions.

Proof of Theorem 4.4.9: Compute the Laplace transform on Eq. (4.4.6),

$$\mathcal{L}[y_n''] + a_1\, \mathcal{L}[y_n'] + a_0\, \mathcal{L}[y_n] = \mathcal{L}[\delta_n(t-c)].$$

Recall the relations between the Laplace transform and derivatives and use the initial conditions,

$$\mathcal{L}[y_n''] = s^2\, \mathcal{L}[y_n] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\, \mathcal{L}[y_n] - y_0,$$

and use these relation in the differential equation,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_n] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[\delta_n(t-c)],$$

Since δ_n satisfies that $\lim_{n\to\infty} \delta_n(t-c) = \delta(t-c)$, an argument like the one in the proof of Theorem 4.4.5 says that for $c \geqslant 0$ holds

$$\mathcal{L}[\delta_n(t-c)] = \mathcal{L}[\delta(t-c)] \quad \Rightarrow \quad \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] = e^{-cs}.$$

Then

$$(s^2 + a_1 s + a_0) \lim_{n\to\infty} \mathcal{L}[y_n] - sy_0 - y_1 - a_1 y_0 = e^{-cs}.$$

Interchanging limits and Laplace transforms we get

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = e^{-cs},$$

which is equivalent to

$$\big(s^2\, \mathcal{L}[y] - sy_0 - y_1\big) + a_1\, \big(s\, \mathcal{L}[y] - y_0\big) - a_0\, \mathcal{L}[y] = e^{-cs}.$$

This establishes the Theorem. □

4.4.6. Exercises.

4.4.1.- ***** Find the solution to the initial value problem

$$y'' - 8y' + 16y = \cos(\pi t)\,\delta(t-1),$$
$$y(0) = 0, \qquad y'(0) = 0.$$

4.4.2.- .

4.5. Convolutions and Solutions

Solutions of initial value problems for linear nonhomogeneous differential equations can be decomposed in a nice way. The part of the solution coming from the initial data can be separated from the part of the solution coming from the nonhomogeneous source function. Furthermore, the latter is a kind of product of two functions, the source function itself and the impulse response function from the differential operator. This kind of product of two functions is the subject of this section. This kind of product is what we call the convolution of two functions.

4.5.1. Definition and Properties.

One can say that the convolution is a generalization of the pointwise product of two functions. In a convolution one multiplies the two functions evaluated at different points and then integrates the result. Here is a precise definition.

Definition 4.5.1. *The* ***convolution*** *of functions f and g is a function $f * g$ given by*

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{4.5.1}$$

Remark: The convolution is defined for functions f and g such that the integral in (4.5.1) is defined. For example for f and g piecewise continuous functions, or one of them continuous and the other a Dirac's delta generalized function.

Example 4.5.1. Find $f * g$ the convolution of the functions $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: The definition of convolution is,

$$(f * g)(t) = \int_0^t e^{-\tau}\sin(t-\tau)\,d\tau.$$

This integral is not difficult to compute. Integrate by parts twice,

$$\int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t - \int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau,$$

that is,

$$2\int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t = e^{-t} - \cos(t) - 0 + \sin(t).$$

We then conclude that

$$(f * g)(t) = \frac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big]. \tag{4.5.2}$$

◁

Example 4.5.2. Graph the convolution of

$$f(\tau) = u(\tau) - u(\tau - 1),$$

$$g(\tau) = \begin{cases} 2\,e^{-2\tau} & \text{for} \quad \tau \geqslant 0 \\ 0 & \text{for} \quad \tau < 0. \end{cases}$$

Solution: Notice that

$$g(-\tau) = \begin{cases} 2\,e^{2\tau} & \text{for} \quad \tau \leqslant 0 \\ 0 & \text{for} \quad \tau > 0. \end{cases}$$

Then we have that

$$g(t-\tau) = g(-(\tau - t)) \begin{cases} 2\,e^{2(\tau - t)} & \text{for} \quad \tau \leqslant t \\ 0 & \text{for} \quad \tau > t. \end{cases}$$

In the graphs below we can see that the values of the convolution function $f * g$ measure the overlap of the functions f and g when one function slides over the other.

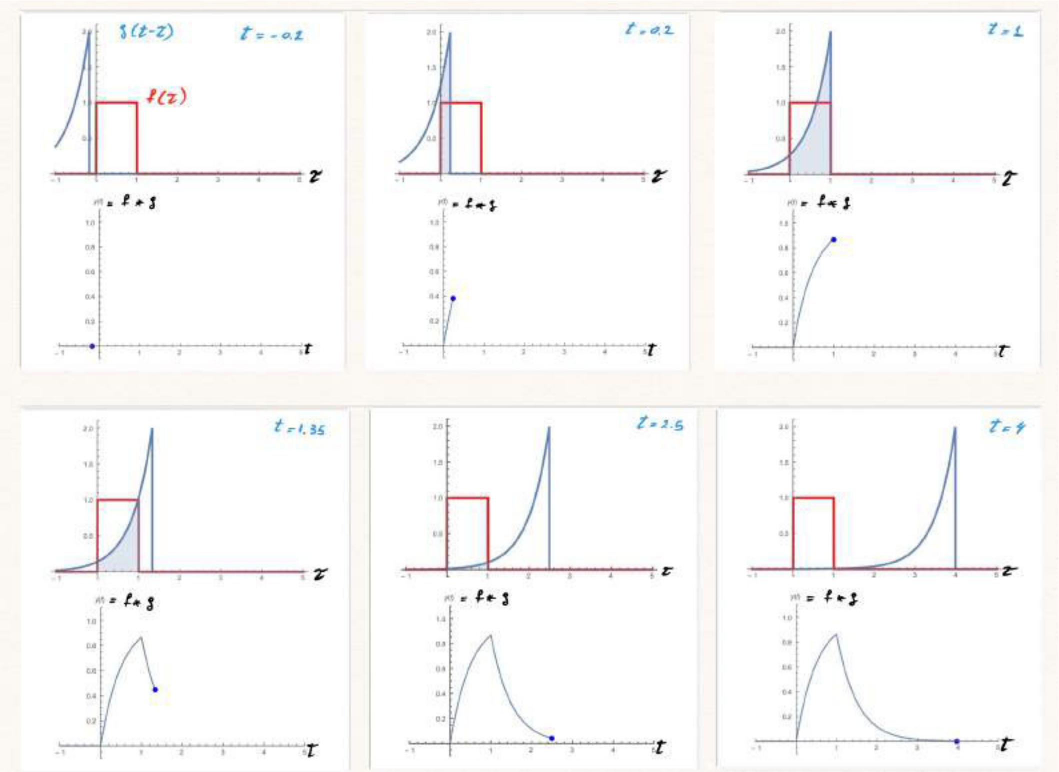


FIGURE 10. The graphs of f, g, and $f * g$.

◁

A few properties of the convolution operation are summarized in the Theorem below. But we save the most important property for the next subsection.

Theorem 4.5.2 (Properties). *For every piecewise continuous functions f, g, and h, hold:*

(i) *Commutativity:* $f * g = g * f$;

(ii) *Associativity:* $f * (g * h) = (f * g) * h$;

(iii) *Distributivity:* $f * (g + h) = f * g + f * h$;

(iv) *Neutral element:* $f * 0 = 0$;

(v) *Identity element:* $f * \delta = f$.

Proof of Theorem 4.5.2: We only prove properties (i) and (v), the rest are left as an exercise and they are not so hard to obtain from the definition of convolution. The first property can be obtained by a change of the integration variable as follows,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Now introduce the change of variables, $\hat{\tau} = t - \tau$, which implies $d\hat{\tau} = -d\tau$, then

$$\begin{aligned}(f * g)(t) &= \int_t^0 f(t-\hat{\tau})\, g(\hat{\tau})(-1)\, d\hat{\tau}\\ &= \int_0^t g(\hat{\tau})\, f(t-\hat{\tau})\, d\hat{\tau},\end{aligned}$$

so we conclude that

$$(f * g)(t) = (g * f)(t).$$

We now move to property (v), which is essentially a property of the Dirac delta,

$$(f * \delta)(t) = \int_0^t f(\tau)\, \delta(t-\tau)\, d\tau = f(t).$$

This establishes the Theorem. □

4.5.2. The Laplace Transform.

The Laplace transform of a convolution of two functions is the pointwise product of their corresponding Laplace transforms. This result will be a key part in the solution decomposition result we show at the end of the section.

Theorem 4.5.3 (Laplace Transform)**.** *If both $\mathcal{L}[g]$ and $\mathcal{L}[g]$ exist, including the case where either f or g is a Dirac's delta, then*

$$\mathcal{L}[f * g] = \mathcal{L}[f]\, \mathcal{L}[g]. \tag{4.5.3}$$

Remark: It is not an accident that the convolution of two functions satisfies Eq. (4.5.3). The definition of convolution is chosen so that it has this property. One can see that this is the case by looking at the proof of Theorem 4.5.3. One starts with the expression $\mathcal{L}[f]\,\mathcal{L}[g]$, then changes the order of integration, and one ends up with the Laplace transform of some quantity. Because this quantity appears in that expression, is that it deserves a name. This is how the convolution operation was created.

Proof of Theorem 4.5.3: We start writing the right hand side of Eq. (4.5.1), the product $\mathcal{L}[f]\,\mathcal{L}[g]$. We write the two integrals coming from the individual Laplace transforms and we rewrite them in an appropriate way.

$$\begin{aligned}\mathcal{L}[f]\,\mathcal{L}[g] &= \Big[\int_0^\infty e^{-st} f(t)\, dt\Big]\, \Big[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\, d\tilde{t}\Big]\\ &= \int_0^\infty e^{-s\tilde{t}} g(\tilde{t}) \Big(\int_0^\infty e^{-st} f(t)\, dt\Big)\, d\tilde{t}\\ &= \int_0^\infty g(\tilde{t}) \Big(\int_0^\infty e^{-s(t+\tilde{t})} f(t)\, dt\Big)\, d\tilde{t},\end{aligned}$$

where we only introduced the integral in t as a constant inside the integral in $\tilde{t}$. Introduce the change of variables in the inside integral $\tau = t + \tilde{t}$, hence $d\tau = dt$. Then, we get

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t})\Big(\int_{\tilde{t}}^\infty e^{-s\tau} f(\tau - \tilde{t})\,d\tau\Big)\,d\tilde{t} \tag{4.5.4}$$

$$= \int_0^\infty \int_{\tilde{t}}^\infty e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\,d\tau\,d\tilde{t}. \tag{4.5.5}$$

Here is the key step. We must switch the order of integration. From Fig. 11 we see that changing the order of integration gives the following expression,

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde{t})\, f(\tau - \tilde{t})\,d\tilde{t}\,d\tau.$$

Then, is straightforward to check that

$$\begin{aligned}\mathcal{L}[f]\,\mathcal{L}[g] &= \int_0^\infty e^{-s\tau}\Big(\int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\,d\tilde{t}\Big)\,d\tau\\ &= \int_0^\infty e^{-s\tau}(g * f)(\tau)\,dt\\ &= \mathcal{L}[g * f] \quad \Rightarrow \quad \mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[f * g].\end{aligned}$$

This establishes the Theorem. □

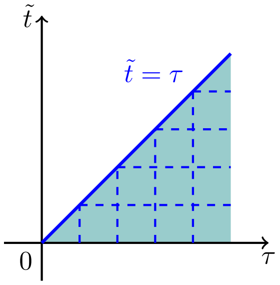


FIGURE 11. Domain of integration in (4.5.5).

Example 4.5.3. Compute the Laplace transform of the function $u(t) = \displaystyle\int_0^t e^{-\tau}\sin(t - \tau)\,d\tau$.

Solution: The function u above is the convolution of the functions

$$f(t) = e^{-t}, \qquad g(t) = \sin(t),$$

that is, $u = f * g$. Therefore, Theorem 4.5.3 says that

$$\mathcal{L}[u] = \mathcal{L}[f * g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Since,

$$\mathcal{L}[f] = \mathcal{L}[e^{-t}] = \frac{1}{s+1}, \qquad \mathcal{L}[g] = \mathcal{L}[\sin(t)] = \frac{1}{s^2+1},$$

we then conclude that $\mathcal{L}[u] = \mathcal{L}[f * g]$ is given by

$$\mathcal{L}[f * g] = \frac{1}{(s+1)(s^2+1)}.$$

◁

Example 4.5.4. Use the Laplace transform to compute $u(t) = \displaystyle\int_0^t e^{-\tau}\sin(t - \tau)\,d\tau$.

Solution: Since $u = f * g$, with $f(t) = e^{-t}$ and $g(t) = \sin(t)$, then from Example 4.5.3,

$$\mathcal{L}[u] = \mathcal{L}[f * g] = \frac{1}{(s+1)(s^2+1)}.$$

A partial fraction decomposition of the right hand side above implies that

$$\begin{aligned}\mathcal{L}[u] &= \frac{1}{2}\Big(\frac{1}{(s+1)} + \frac{(1-s)}{(s^2+1)}\Big)\\ &= \frac{1}{2}\Big(\frac{1}{(s+1)} + \frac{1}{(s^2+1)} - \frac{s}{(s^2+1)}\Big)\\ &= \frac{1}{2}\Big(\mathcal{L}[e^{-t}] + \mathcal{L}[\sin(t)] - \mathcal{L}[\cos(t)]\Big).\end{aligned}$$

This says that

$$u(t) = \frac{1}{2}\big(e^{-t} + \sin(t) - \cos(t)\big).$$

So, we recover Eq. (4.5.2) in Example 4.5.1, that is,

$$(f * g)(t) = \frac{1}{2}\big(e^{-t} + \sin(t) - \cos(t)\big),$$

◁

Example 4.5.5. Find the function g such that $f(t) = \int_0^t \sin(4\tau)\, g(t-\tau)\, d\tau$ has the Laplace transform $\mathcal{L}[f] = \dfrac{s}{(s^2+16)((s-1)^2+9)}$.

Solution: Since $f(t) = \sin(4t) * g(t)$, we can write

$$\begin{aligned}\frac{s}{(s^2+16)((s-1)^2+9)} = \mathcal{L}[f] &= \mathcal{L}[\sin(4t) * g(t)]\\ &= \mathcal{L}[\sin(4t)]\,\mathcal{L}[g]\\ &= \frac{4}{(s^2+4^2)}\,\mathcal{L}[g],\end{aligned}$$

so we get that

$$\frac{4}{(s^2+4^2)}\,\mathcal{L}[g] = \frac{s}{(s^2+16)((s-1)^2+9)} \quad\Rightarrow\quad \mathcal{L}[g] = \frac{1}{4}\,\frac{s}{(s-1)^2+3^2}.$$

We now rewrite the right-hand side of the last equation,

$$\mathcal{L}[g] = \frac{1}{4}\,\frac{(s-1+1)}{(s-1)^2+3^2} \quad\Rightarrow\quad \mathcal{L}[g] = \frac{1}{4}\,\Big(\frac{(s-1)}{(s-1)^2+3^2} + \frac{1}{3}\,\frac{3}{(s-1)^2+3^2}\Big),$$

that is,

$$\mathcal{L}[g] = \frac{1}{4}\,\Big(\mathcal{L}[\cos(3t)](s-1) + \frac{1}{3}\mathcal{L}[\sin(3t)](s-1)\Big) = \frac{1}{4}\,\Big(\mathcal{L}[e^t\,\cos(3t)] + \frac{1}{3}\,\mathcal{L}[e^t\,\sin(3t)]\Big),$$

which leads us to

$$g(t) = \frac{1}{4}\,e^t\,\Big(\cos(3t) + \frac{1}{3}\,\sin(3t)\Big)$$

◁

4.5.3. Solution Decomposition. The Solution Decomposition Theorem is the main result of this section. Theorem 4.5.4 shows one way to write the solution to a general initial value problem for a linear second order differential equation with constant coefficients. The solution to such problem can always be divided in two terms. The first term contains information only about the initial data. The second term contains information only about the source function. This second term is a convolution of the source function itself and the impulse response function of the differential operator.

Theorem 4.5.4 (Solution Decomposition)**.** *Given constants* a_0, a_1, y_0, y_1 *and a piecewise continuous function* g*, the solution* y *to the initial value problem*

$$y'' + a_1\, y' + a_0\, y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1, \tag{4.5.6}$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta * g)(t), \tag{4.5.7}$$

where y_h *is the solution of the homogeneous initial value problem*

$$y_h'' + a_1\, y_h' + a_0\, y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1, \tag{4.5.8}$$

and y_δ *is the impulse response solution, that is,*

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Remark: The solution decomposition in Eq. (4.5.7) can be written in the equivalent way

$$y(t) = y_h(t) + \int_0^t y_\delta(\tau) g(t-\tau)\, d\tau.$$

Also, recall that the impulse response function can be written in the equivalent way

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big], \quad c \neq 0, \quad \text{and} \quad y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big], \quad c = 0.$$

Proof of Theorem4.5.4: Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y''] + a_1\, \mathcal{L}[y'] + a_0\, \mathcal{L}[y] = \mathcal{L}[g(t)].$$

Recalling the relations between Laplace transforms and derivatives,

$$\mathcal{L}[y''] = s^2\, \mathcal{L}[y] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\, \mathcal{L}[y] - y_0.$$

we re-write the differential equation for y as an algebraic equation for $\mathcal{L}[y]$,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

As usual, it is simple to solve the algebraic equation for $\mathcal{L}[y]$,

$$\mathcal{L}[y] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\, \mathcal{L}[g(t)].$$

Now, the function y_h is the solution of Eq. (4.5.8), that is,

$$\mathcal{L}[y_h] = \frac{(s + a_1)y_0 + y_1}{(s^2 + a_1 s + a_0)}.$$

And by the definition of the impulse response solution y_δ we have that

$$\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}.$$

These last three equation imply,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\, \mathcal{L}[g(t)].$$

This is the Laplace transform version of Eq. (4.5.7). Inverting the Laplace transform above,

$$y(t) = y_h(t) + \mathcal{L}^{-1}\big[\mathcal{L}[y_\delta]\, \mathcal{L}[g(t)]\big].$$

Using the result in Theorem 4.5.3 in the last term above we conclude that

$$y(t) = y_h(t) + (y_\delta * g)(t).$$

□

Example 4.5.6. Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = g(t), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: We first find the impuse response function

$$y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big], \qquad p(s) = s^2 + 2s + 2.$$

since p has complex roots, we complete the square,

$$s^2 + 2s + 2 = s^2 + 2s + 1 - 1 + 2 = (s+1)^2 + 1,$$

so we get

$$y_\delta(t) = \mathcal{L}^{-1}\Big[\frac{1}{(s+1)^2 + 1}\Big] \quad \Rightarrow \quad y_\delta(t) = e^{-t}\,\sin(t).$$

We now compute the solution to the homogeneous problem

$$y_h'' + 2\,y_h' + 2\,y_h = 0, \qquad y_h(0) = 1, \quad y_h'(0) = -1.$$

Using Laplace transforms we get

$$\mathcal{L}[y_h''] + 2\,\mathcal{L}[y_h'] + 2\,\mathcal{L}[y_h] = 0,$$

and recalling the relations between the Laplace transform and derivatives,

$$\big(s^2\,\mathcal{L}[y_h] - s\,y_h(0) - y_h'(0)\big) + 2\big(\mathcal{L}[y_h'] = s\,\mathcal{L}[y_h] - y_h(0)\big) + 2\mathcal{L}[y_h] = 0,$$

using our initial conditions we get $(s^2 + 2s + 2)\,\mathcal{L}[y_h] - s + 1 - 2 = 0$, so

$$\mathcal{L}[y_h] = \frac{(s+1)}{(s^2 + 2s + 2)} = \frac{(s+1)}{(s+1)^2 + 1},$$

so we obtain

$$y_h(t) = \mathcal{L}\Big[e^{-t}\,\cos(t)\Big].$$

Therefore, the solution to the original initial value problem is

$$y(t) = y_h(t) + (y_\delta * g)(t) \quad \Rightarrow \quad y(t) = e^{-t}\,\cos(t) + \int_0^t e^{-\tau}\,\sin(\tau)\,g(t-\tau)\,d\tau.$$

◁

Example 4.5.7. Use the Laplace transform to solve the same IVP as above.

$$y'' + 2\,y' + 2\,y = g(t), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Compute the Laplace transform of the differential equation above,

$$\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[g(t)],$$

and recall the relations between the Laplace transform and derivatives,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy(0) - y'(0), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0).$$

Introduce the initial conditions in the equation above,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1,$$

and these two equation into the differential equation,

$$(s^2 + 2s + 2)\,\mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[g(t)].$$

Reorder terms to get

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2 + 2s + 2)} + \frac{1}{(s^2 + 2s + 2)}\,\mathcal{L}[g(t)].$$

Now, the function y_h is the solution of the homogeneous initial value problem with the same initial conditions as y, that is,

$$\mathcal{L}[y_h] = \frac{(s+1)}{(s^2+2s+2)} = \frac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)].$$

Now, the function y_δ is the impulse response solution for the differential equation in this Example, that is,

$$cL[y_\delta] = \frac{1}{(s^2+2s+2)} = \frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)].$$

If we put all this information together and we get

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t),$$

More explicitly, we get

$$y(t) = e^{-t}\,\cos(t) + \int_0^t e^{-\tau}\sin(\tau)\,g(t-\tau)\,d\tau.$$

$\lhd$

4.5.4. Exercises.

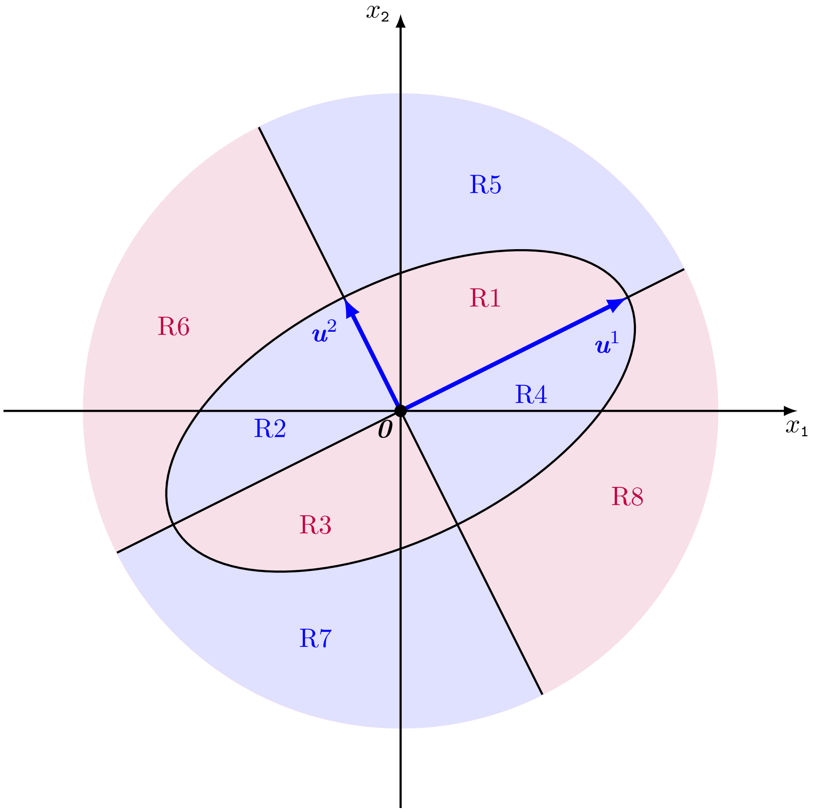
4.5.1.- .

4.5.2.- .

CHAPTER 5

Systems of Linear Differential Equations

Newton's second law of motion for point particles is one of the first differential equations ever written. Even this early example of a differential equation consists not of a single equation but of a system of three equation on three unknowns. The unknown functions are the particle three coordinates in space as function of time. One important difficulty to solve a differential system is that the equations in a system are usually coupled. One cannot solve for one unknown function without knowing the other unknowns. In this chapter we study how to solve the system in the particular case that the equations can be uncoupled. We call such systems diagonalizable. Explicit formulas for the solutions can be written in this case. Later we generalize this idea to systems that cannot be uncoupled.



5.1. General Properties

This Section is a generalization of the ideas in § 2.1 from a single equation to a system of equations. We start introducing a linear system of differential equations with variable coefficients and the associated initial value problem. We show that such initial value problems always have a unique solution. We then introduce the concepts of fundamental solutions, general solution, fundamental matrix, the Wronskian, and Abel's Theorem for systems. We assume that the reader is familiar with the concepts of linear algebra given in Chapter 8.

5.1.1. First Order Linear Systems.

A single differential equation on one unknown function is often not enough to describe certain physical problems. For example problems in several dimensions or containing several interacting particles. The description of a point particle moving in space under Newton's law of motion requires three functions of time—the space coordinates of the particle—to describe the motion together with three differential equations. To describe several proteins activating and deactivating each other inside a cell also requires as many unknown functions and equations as proteins in the system. In this section we present a first step aimed to describe such physical systems. We start introducing a first order linear differential system of equations.

Definition 5.1.1. *An $n \times n$* ***first order linear differential system*** *is the equation*

$$\boldsymbol{x}'(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{b}(t), \tag{5.1.1}$$

where the $n \times n$ coefficient matrix A, the source n-vector $\boldsymbol{b}$, and the unknown n-vector $\boldsymbol{x}$ are given in components by

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \qquad \boldsymbol{b}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

The system in 5.1.1 is called ***homogeneous*** *iff the source vector $\boldsymbol{b} = \boldsymbol{0}$, of* ***constant coefficients*** *iff the matrix A is constant, and* ***diagonalizable*** *iff the matrix A is diagonalizable.*

Remarks:

(a) The derivative of a a vector valued function is defined as $\boldsymbol{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$.

(b) By the definition of the matrix-vector product, Eq. (5.1.1) can be written as

$$\begin{aligned} x_1'(t) &= a_{11}(t)\,x_1(t) + \cdots + a_{1n}(t)\,x_n(t) + b_1(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)\,x_1(t) + \cdots + a_{nn}(t)\,x_n(t) + b_n(t). \end{aligned}$$

(c) We recall that in § 8.3 we say that a square matrix A is diagonalizable iff there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

A *solution* of an $n \times n$ linear differential system is an n-vector valued function $\boldsymbol{x}$, that is, a set of n functions $\{x_1, \cdots, x_n\}$, that satisfy every differential equation in the system. When we write down the equations we will usually write $\boldsymbol{x}$ instead of $\boldsymbol{x}(t)$.

Example 5.1.1. The case $n = 1$ is a single differential equation: Find a solution x_1 of

$$x_1' = a_{11}(t)\,x_1 + b_1(t).$$

Solution: This is a linear first order equation, and solutions can be found with the integrating factor method described in Section 1.2. ◁

Example 5.1.2. Find the coefficient matrix, the source vector and the unknown vector for the 2×2 linear system

$$\begin{aligned} x_1' &= a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + g_1(t),\\ x_2' &= a_{21}(t)\,x_1 + a_{22}(t)\,x_2 + g_2(t). \end{aligned}$$

Solution: The coefficient matrix A, the source vector $\boldsymbol{b}$, and the unknown vector $\boldsymbol{x}$ are,

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \qquad \boldsymbol{b}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

◁

Example 5.1.3. Use matrix notation to write down the 2×2 system given by

$$\begin{aligned} x_1' &= x_1 - x_2,\\ x_2' &= -x_1 + x_2. \end{aligned}$$

Solution: In this case, the matrix of coefficients and the unknown vector have the form

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

This is an homogeneous system, so the source vector $\boldsymbol{b} = \boldsymbol{0}$. The differential equation can be written as follows,

$$\begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= -x_1 + x_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}' = A\boldsymbol{x}.$$

◁

Example 5.1.4. Find the explicit expression for the linear system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \boldsymbol{b}(t) = \begin{bmatrix} e^{t} \\ 2e^{3t} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{t} \\ 2e^{3t} \end{bmatrix}, \quad \Leftrightarrow \quad \begin{aligned} x_1' &= x_1 + 3x_2 + e^{t}, \\ x_2' &= 3x_1 + x_2 + 2e^{3t}. \end{aligned}$$

◁

Example 5.1.5. Show that the vector valued functions $\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$ are solutions to the 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

Solution: We compute the left-hand side and the right-hand side of the differential equation above for the function $\boldsymbol{x}^{(1)}$ and we see that both side match, that is,

$$A\boldsymbol{x}^{(1)} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}; \quad \boldsymbol{x}^{(1)\prime} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(e^{2t}\right)' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} 2\,e^{2t},$$

so we conclude that $\boldsymbol{x}^{(1)\,\prime} = A\boldsymbol{x}^{(1)}$. Analogously,

$$A\boldsymbol{x}^{(2)} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^{-t} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}; \quad \boldsymbol{x}^{(2)\,\prime} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(e^{-t}\right)' = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t},$$

so we conclude that $\boldsymbol{x}^{(2)\,\prime} = A\boldsymbol{x}^{(2)}$. ◁

Example 5.1.6. Find the explicit expression of the most general 3×3 homogeneous linear differential system.

Solution: This is a system of the form $\boldsymbol{x}' = A(t)\,\boldsymbol{x}$, with A being a 3×3 matrix. Therefore, we need to find functions x_1, x_2, and x_3 solutions of

$$\begin{aligned} x_1' &= a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + a_{13}(t)\,x_3 \\ x_2' &= a_{21}(t)\,x_1 + a_{22}(t)\,x_2 + a_{13}(t)\,x_3 \\ x_3' &= a_{31}(t)\,x_1 + a_{32}(t)\,x_2 + a_{33}(t)\,x_3. \end{aligned}$$

◁

5.1.2. Existence of Solutions.

We first introduce the initial value problem for linear differential equations. This problem is similar to initial value problem for a single differential equation. In the case of an $n \times n$ first order system we need n initial conditions, one for each unknown function, which are collected in an n-vector.

Definition 5.1.2. *An* ***Initial Value Problem*** *for an $n \times n$ linear differential system is the following: Given an $n \times n$ matrix valued function A, and an n-vector valued function $\boldsymbol{b}$, a real constant t_0, and an n-vector $\boldsymbol{x}_0$, find an n-vector valued function $\boldsymbol{x}$ solution of*

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x} + \boldsymbol{b}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

Remark: The initial condition vector $\boldsymbol{x}_0$ represents n conditions, one for each component of the unknown vector $\boldsymbol{x}$.

Example 5.1.7. Write down explicitly the initial value problem for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given by

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: This is a 2×2 system in the unknowns x_1, x_2, with two linear equations

$$\begin{aligned} x_1' &= x_1 + 3x_2 \\ x_2' &= 3x_1 + x_2, \end{aligned}$$

and the initial conditions $x_1(0) = 2$ and $x_2(0) = 3$. ◁

The main result about existence and uniqueness of solutions to an initial value problem for a linear system is also analogous to Theorem 2.1.2

Theorem 5.1.3 (Existence and Uniqueness)**.** *If the functions A and $\boldsymbol{b}$ are continuous on an open interval $I \subset \mathbb{R}$, and if $\boldsymbol{x}_0$ is any constant vector and t_0 is any constant in I, then there exist only one function $\boldsymbol{x}$, defined an interval $\tilde{I} \subset I$ with $t_0 \in \tilde{I}$, solution of the initial value problem*

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x} + \boldsymbol{b}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{5.1.2}$$

Remark: The fixed point argument used in the proof of Picard-Lindelöf's Theorem 1.6.2 can be extended to prove Theorem 5.1.3. This proof will be presented later on.

5.1.3. Order Transformations.

There is a relation between solutions to $n \times n$ systems of linear differential equations and the solutions of n-th order linear scalar differential equations. This relation can take different forms. In this section we focus on the case of $n = 2$ and we show two of these relations: the first order reduction and the second order reduction.

It is useful to have a correspondence between solutions of an $n \times n$ linear system and an n-th order scalar equation. One reason is that concepts developed for one of the equations can be translated to the other equation. For example, we have introduced several concepts when we studied 2-nd order scalar linear equations in § 2.1, concepts such as the superposition property, fundamental solutions, general solutions, the Wronskian, and Abel's theorem. It turns out that these concepts can be translated to 2×2 (and in general to $n \times n$) linear differential systems.

Theorem 5.1.4 (First Order Reduction). *A function y solves the second order equation*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \tag{5.1.3}$$

iff the functions $x_1 = y$ and $x_2 = y'$ are solutions to the 2×2 first order differential system

$$x_1' = x_2, \tag{5.1.4}$$

$$x_2' = -a_0(t)\, x_1 - a_1(t)\, x_2 + b(t). \tag{5.1.5}$$

Proof of Theorem 5.1.4:
($\Rightarrow$) Given a solution y of Eq. (5.1.3), introduce the functions $x_1 = y$ and $x_2 = y'$. Therefore Eq. (5.1.4) holds, due to the relation

$$x_1' = y' = x_2,$$

Also Eq. (5.1.5) holds, because of the equation

$$x_2' = y'' = -a_0(t)\, y - a_1(t)\, y' + b(t) \quad \Rightarrow \quad x_2'' = -a_0(t)\, x_1 - a_1(t)\, x_2 + b(t).$$

($\Leftarrow$) Differentiate Eq. (5.1.4) and introduce the result into Eq. (5.1.5), that is,

$$x_1'' = x_2' \quad \Rightarrow \quad x_1'' = -a_0(t)\, x_1 - a_1(t)\, x_1' + b(t).$$

Denoting $y = x_1$, we obtain,

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t).$$

This establishes the Theorem. □

Example 5.1.8. Express as a first order system the second order equation

$$y'' + 2y' + 2y = \sin(at).$$

Solution: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

We conclude that

$$x_1' = x_2, \qquad x_2' = -2x_1 - 2x_2 + \sin(at).$$

◁

Remark: The transformation in Theorem 5.1.4 can be generalized to $n\times n$ linear differential systems and n-th order scalar linear equations, where $n \geqslant 2$.

We now introduce a second relation between systems and scalar equations.

Theorem 5.1.5 (Second Order Reduction)**.** *Any* 2×2 *constant coefficients linear system* $\boldsymbol{x}' = A\,\boldsymbol{x}$, *with* $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, *can be written as second order equations for* x_1 *and* x_2,

$$\boldsymbol{x}'' - \operatorname{tr}(A)\,\boldsymbol{x}' + \det(A)\,\boldsymbol{x} = \boldsymbol{0}. \tag{5.1.6}$$

Furthermore, the solution to the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}$, *with* $\boldsymbol{x}(0) = \boldsymbol{x}_0$, *also solves the initial value problem given by Eq.* (5.1.6) *with initial condition*

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad \boldsymbol{x}'(0) = A\boldsymbol{x}_0. \tag{5.1.7}$$

Remark: In components, Eq. (5.1.6) has the form

$$x_1'' - \operatorname{tr}(A)\,x_1' + \det(A)\,x_1 = 0, \tag{5.1.8}$$

$$x_2'' - \operatorname{tr}(A)\,x_2' + \det(A)\,x_2 = 0. \tag{5.1.9}$$

First Proof of Theorem 5.1.5: We start with the following identity, which is satisfied by every 2×2 matrix A, (exercise: prove it on 2×2 matrices by a straightforward calculation)

$$A^2 - \operatorname{tr}(A)\,A + \det(A)\,I = 0.$$

This identity is the particular case $n = 2$ of the Cayley-Hamilton Theorem, which holds for every $n\times n$ matrix. If we use this identity on the equation for $\boldsymbol{x}''$ we get the equation in Theorem 5.1.5, because

$$\boldsymbol{x}'' = \big(A\,\boldsymbol{x}\big)' = A\,\boldsymbol{x}' = A^2\,\boldsymbol{x} = \operatorname{tr}(A)\,A\boldsymbol{x} - \det(A)I\boldsymbol{x}.$$

Recalling that $A\,\boldsymbol{x} = \boldsymbol{x}'$, and $I\boldsymbol{x} = \boldsymbol{x}$, we get the vector equation

$$\boldsymbol{x}'' - \operatorname{tr}(A)\,\boldsymbol{x}' + \det(A)\,\boldsymbol{x} = \boldsymbol{0}.$$

The initial conditions for a second order differential equation are $\boldsymbol{x}(0)$ and $\boldsymbol{x}'(0)$. The first condition is given by hypothesis, $\boldsymbol{x}(0) = \boldsymbol{x}_0$. The second condition comes from the original first order system evaluated at $t = 0$, that is $\boldsymbol{x}'(0) = A\boldsymbol{x}(0) = A\boldsymbol{x}_0$. This establishes the Theorem. ☐

Second Proof of Theorem 5.1.5: This proof is based on a straightforward computation. Denote $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the system has the form

$$x_1' = a_{11}\,x_1 + a_{12}\,x_2 \tag{5.1.10}$$

$$x_2' = a_{21}\,x_1 + a_{22}\,x_2. \tag{5.1.11}$$

We start considering the case $a_{12} \neq 0$. Compute the derivative of the first equation,

$$x_1'' = a_{11}\,x_1' + a_{12}\,x_2'.$$

Use Eq. (5.1.11) to replace x_2' on the right-hand side above,

$$x_1'' = a_{11}\,x_1' + a_{12}\big(a_{21}\,x_1 + a_{22}\,x_2\big).$$

Since we are assuming that $a_{12} \neq 0$, we can replace the term with x_2 above using Eq. (5.1.10),

$$x_1'' = a_{11}\,x_1' + a_{12}a_{21}\,x_1 + a_{12}a_{22}\,\frac{\big(x_1' - a_{11}\,x_1\big)}{a_{12}}.$$

A simple cancellation and reorganization of terms gives the equation,

$$x_1'' = (a_{11} + a_{22})\, x_1' + (a_{12}a_{21} - a_{11}a_{22})\, x_1.$$

Recalling that $\operatorname{tr}(A) = a_{11} + a_{22}$, and $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, we get

$$x_1'' - \operatorname{tr}(A)\, x_1' + \det(A)\, x_1 = 0.$$

The initial conditions for x_1 are $x_1(0)$ and $x_1'(0)$. The first one comes from the first component of $\boldsymbol{x}(0) = \boldsymbol{x}_0$, that is

$$x_1(0) = x_{01}. \tag{5.1.12}$$

The second condition comes from the first component of the first order differential equation evaluated at $t = 0$, that is $\boldsymbol{x}'(0) = A\boldsymbol{x}(0) = A\boldsymbol{x}_0$. The first component is

$$x_1'(0) = a_{11}\, x_{01} + a_{12}\, x_{02}. \tag{5.1.13}$$

Consider now the case $a_{12} = 0$. In this case the system is

$$\begin{aligned} x_1' &= a_{11}\, x_1 \\ x_2' &= a_{21}\, x_1 + a_{22}\, x_2. \end{aligned}$$

In this case compute one more derivative in the first equation above,

$$x_1'' = a_{11}\, x_1'.$$

Now rewrite the first equation in the system as follows

$$a_{22}\,(-x_1' + a_{11}\, x_1) = 0.$$

Adding these last two equations for x_1 we get

$$x_1'' - a_{11}\, x_1' + a_{22}\,(-x_1' + a_{11}\, x_1) = 0,$$

So we get the equation

$$x_1'' - (a_{11} + a_{22})\, x_1' + (a_{11}a_{22})\, x_1 = 0.$$

Recalling that in the case $a_{12} = 0$ we have $\operatorname{tr}(A) = a_{11} + a_{22}$, and $\det(A) = a_{11}a_{22}$, we get

$$x_1'' - \operatorname{tr}(A)\, x_1' + \det(A)\, x_1 = 0.$$

The initial conditions are the same as in the case $a_{12} \neq 0$. A similar calculation gives x_2 and its initial conditions. This establishes the Theorem. ☐

Example 5.1.9. Express as a single second order equation the 2×2 system and solve it,

$$\begin{aligned} x_1' &= -x_1 + 3x_2, \\ x_2' &= x_1 - x_2. \end{aligned}$$

Solution: Instead of using the result from Theorem 5.1.5, we solve this problem following the second proof of that theorem. But instead of working with x_1, we work with x_2. We start computing x_1 from the second equation: $x_1 = x_2' + x_2$. We then introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2 \quad \Rightarrow \quad x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

so we obtain the second order equation

$$x_2'' + 2x_2' - 2x_2 = 0.$$

We solve this equation with the methods studied in Chapter 2, that is, we look for solutions of the form $x_2(t) = e^{rt}$, with r solution of the characteristic equation

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_\pm = -1 \pm \sqrt{3}.$$

Therefore, the general solution to the second order equation above is

$$x_2 = c_+ \, e^{(1+\sqrt{3})t} + c_- \, e^{(1-\sqrt{3})t}, \qquad c_+, \; c_- \in \mathbb{R}.$$

Since x_1 satisfies the same equation as x_2, we obtain the same general solution

$$x_1 = \tilde{c}_+ \, e^{(1+\sqrt{3})t} + \tilde{c}_- \, e^{(1-\sqrt{3})t}, \qquad \tilde{c}_+, \; \tilde{c}_- \in \mathbb{R}.$$

◁

Example 5.1.10. Write the first order initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix},$$

as a second order initial value problem for x_1. Repeat the calculations for x_2.

Solution: From Theorem 5.1.5 we know that both x_1 and x_2 satisfy the same differential equation. Since $\operatorname{tr}(A) = 1 + 4 = 5$ and $\det(A) = 4 - 6 = -2$, the differential equations are

$$x_1'' - 5\,x_1' - 2\,x_1 = 0, \qquad x_2'' - 5\,x_2' - 2\,x_2 = 0.$$

From the same Theorem we know that the initial conditions for the second order differential equations above are $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\boldsymbol{x}'(0) = A\,\boldsymbol{x}_0$, that is

$$\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad \boldsymbol{x}'(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix},$$

therefore, the initial conditions for x_1 and x_2 are

$$x_1(0) = 5, \quad x_1'(0) = 17, \qquad \text{and} \qquad x_2(0) = 6, \quad x_2'(0) = 39.$$

◁

5.1.4. Homogeneous Systems.

Solutions to a linear homogeneous differential system satisfy the superposition property: Given two solutions of the homogeneous system, their linear combination is also a solution to that system.

Theorem 5.1.6 (Superposition)**.** *If the vector functions $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ are solutions of*

$$\boldsymbol{x}^{(1)\prime} = A\boldsymbol{x}^{(1)}, \qquad \boldsymbol{x}^{(2)\prime} = A\,\boldsymbol{x}^{(2)},$$

then the linear combination $\boldsymbol{x} = a\,\boldsymbol{x}^{(1)} + b\,\boldsymbol{x}^{(2)}$, for all $a, b \in \mathbb{R}$, is also solution of

$$\boldsymbol{x}' = A\,\boldsymbol{x}.$$

Remark: This Theorem contains two particular cases:

(a) $a = b = 1$: If $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions of an homogeneous linear system, so is $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$.
(b) $b = 0$ and a arbitrary: If $\boldsymbol{x}^{(1)}$ is a solution of an homogeneous linear system, so is $a\boldsymbol{x}^{(1)}$.

Proof of Theorem 5.1.6: We check that the function $\boldsymbol{x} = a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}$ is a solution of the differential equation in the Theorem. Indeed, since the derivative of a vector valued function is a linear operation, we get

$$\boldsymbol{x}' = \big(a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}\big)' = a\,\boldsymbol{x}^{(1)\,\prime} + b\,\boldsymbol{x}^{(2)\,\prime}.$$

Replacing the differential equation on the right-hand side above,

$$\boldsymbol{x}' = a\,A\boldsymbol{x}^{(1)} + b\,A\boldsymbol{x}^{(2)}.$$

The matrix-vector product is a linear operation, $A\big(a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}\big) = a\,A\boldsymbol{x}^{(1)} + b\,A\boldsymbol{x}^{(2)}$, hence,

$$\boldsymbol{x}' = A\big(a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}\big) \quad \Rightarrow \quad \boldsymbol{x}' = A\boldsymbol{x}.$$

This establishes the Theorem. □

Example 5.1.11. Verify that $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$ and $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$ are solutions to the homogeneous linear system

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}.$$

Solution: The function $\boldsymbol{x}^{(1)}$ is solution to the differential equation, since

$$\boldsymbol{x}^{(1)\,\prime} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}, \qquad A\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} e^{-2t} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}.$$

We then conclude that $\boldsymbol{x}^{(1)\,\prime} = A\boldsymbol{x}^{(1)}$. Analogously, the function $\boldsymbol{x}^{(2)}$ is solution to the differential equation, since

$$\boldsymbol{x}^{(2)\,\prime} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}, \qquad A\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} e^{4t} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}.$$

We then conclude that $\boldsymbol{x}^{(2)\,\prime} = A\boldsymbol{x}^{(2)}$. To show that $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$ is also a solution we could use the linearity of the matrix-vector product, as we did in the proof of the Theorem 5.1.6. Here we choose the straightforward, although more obscure, calculation: On the one hand,

$$\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)} = \begin{bmatrix} e^{-2t} - e^{4t} \\ e^{-2t} + e^{4t} \end{bmatrix} \quad \Rightarrow \quad \big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big)' = \begin{bmatrix} -2e^{-2t} - 4e^{4t} \\ -2e^{-2t} + 4e^{4t} \end{bmatrix}.$$

On the other hand,

$$A\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big) = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} - e^{4t} \\ e^{-2t} + e^{4t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - e^{4t} - 3e^{-2t} - 3e^{4t} \\ -3e^{-2t} + 3e^{4t} + e^{-2t} + e^{4t} \end{bmatrix},$$

that is,

$$A\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big) = \begin{bmatrix} -2e^{-2t} - 4e^{4t} \\ -2e^{-2t} + 4e^{4t} \end{bmatrix}.$$

We conclude that $\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big)' = A\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big)$. ◁

We now introduce the notion of a linearly dependent and independent set of functions.

Definition 5.1.7. *A set of n vector valued functions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is called **linearly dependent** on an interval $I \in \mathbb{R}$ iff for all $t \in I$ there exist constants $c_1, \cdots, c_n$, not all of them zero, such that it holds*

$$c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t) = \boldsymbol{0}.$$

*A set of n vector valued functions is called **linearly independent** on I iff the set is not linearly dependent.*

Remark: This notion is a generalization of Def. 2.1.6 from two functions to n vector valued functions. For every value of $t \in \mathbb{R}$ this definition agrees with the definition of a set of linearly dependent vectors given in Linear Algebra, reviewed in Chapter 8.

We now generalize Theorem 2.1.7 to linear systems. If you know a linearly independent set of n solutions to an $n \times n$ first order, linear, homogeneous system, then you actually know all possible solutions to that system, since any other solution is just a linear combination of the previous n solutions.

Theorem 5.1.8 (General Solution). *If $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a linearly independent set of solutions of the $n \times n$ system $\boldsymbol{x}' = A\,\boldsymbol{x}$, where A is a continuous matrix valued function, then there exist unique constants $c_1, \cdots, c_n$ such that every solution $\boldsymbol{x}$ of the differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ can be written as the linear combination*

$$\boldsymbol{x}(t) = c_1\,\boldsymbol{x}^{(1)}(t) + \cdots + c_n\,\boldsymbol{x}^{(n)}(t). \tag{5.1.14}$$

Before we present a sketch of the proof for Theorem 5.1.8, it is convenient to state the following the definitions, which come out naturally from Theorem 5.1.8.

Definition 5.1.9.

*(a) The set of functions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a **fundamental set of solutions** of the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ iff the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is linearly independent and $\boldsymbol{x}^{(i)\prime} = A\,\boldsymbol{x}^{(i)}$, for every $i = 1, \cdots, n$.*

*(b) The **general solution** of the homogeneous equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ denotes any vector valued function $\boldsymbol{x}_{\text{gen}}$ that can be written as a linear combination*

$$\boldsymbol{x}_{\text{gen}}(t) = c_1\,\boldsymbol{x}^{(1)}(t) + \cdots + c_n\,\boldsymbol{x}^{(n)}(t),$$

where $\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}$ are the functions in any fundamental set of solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$, while $c_1, \cdots, c_n$ are arbitrary constants.

Remark: The names above are appropriate, since Theorem 5.1.8 says that knowing the n functions of a fundamental set of solutions is equivalent to knowing all solutions to the homogeneous linear differential system.

Example 5.1.12. Show that the set of functions $\left\{\boldsymbol{x}^{(1)} = \begin{bmatrix}1\\1\end{bmatrix} e^{-2t},\ \boldsymbol{x}^{(2)} = \begin{bmatrix}-1\\1\end{bmatrix} e^{4t}\right\}$ is a fundamental set of solutions to the system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix}1 & -3\\ -3 & 1\end{bmatrix}$.

Solution: In Example 5.1.11 we have shown that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions to the differential equation above. We only need to show that these two functions form a linearly independent set. That is, we need to show that the only constants c_1, c_2 solutions of the equation below, for all $t \in \mathbb{R}$, are $c_1 = c_2 = 0$, where

$$\boldsymbol{0} = c_1\,\boldsymbol{x}^{(1)} + c_2\,\boldsymbol{x}^{(2)} = c_1 \begin{bmatrix}1\\1\end{bmatrix} e^{-2t} + c_2 \begin{bmatrix}-1\\1\end{bmatrix} e^{4t} = \begin{bmatrix}e^{-2t} & -e^{4t}\\ e^{-2t} & e^{4t}\end{bmatrix} \begin{bmatrix}c_1\\c_2\end{bmatrix} = X(t)\,\boldsymbol{c},$$

where $X(t) = \big[\boldsymbol{x}^{(1)}(t),\ \boldsymbol{x}^{(2)}(t)\big]$ and $\boldsymbol{c} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$. Using this matrix notation, the linear system for c_1, c_2 has the form

$$X(t)\,\boldsymbol{c} = \boldsymbol{0}.$$

We now show that matrix $X(t)$ is invertible for all $t \in \mathbb{R}$. This is the case, since its determinant is

$$\det\big(X(t)\big) = \begin{vmatrix}e^{-2t} & -e^{4t}\\ e^{-2t} & e^{4t}\end{vmatrix} = e^{2t} + e^{2t} = 2\,e^{2t} \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

Since $X(t)$ is invertible for $t \in \mathbb{R}$, the only solution for the linear system above is $\boldsymbol{c} = \boldsymbol{0}$, that is, $c_1 = c_2 = 0$. We conclude that the set $\big\{\boldsymbol{x}^{(1)},\, \boldsymbol{x}^{(2)}\big\}$ is linearly independent, so it is a fundamental set of solution to the differential equation above. ◁

Proof of Theorem 5.1.8: The superposition property in Theorem 5.1.6 says that given any set of solutions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ of the differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$, the linear combination $\boldsymbol{x}(t) = c_1\,\boldsymbol{x}^{(1)}(t) + \cdots + c_n\,\boldsymbol{x}^{(n)}(t)$ is also a solution. We now must prove that, in the case that $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is linearly independent, every solution of the differential equation is included in this linear combination.

Let $\boldsymbol{x}$ be any solution of the differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$. The uniqueness statement in Theorem 5.1.3 implies that this is the only solution that at t_0 takes the value $\boldsymbol{x}(t_0)$. This means that the initial data $\boldsymbol{x}(t_0)$ parametrizes all solutions to the differential equation. We now try to find the constants $\{c_1, \cdots, c_n\}$ solutions of the algebraic linear system

$$\boldsymbol{x}(t_0) = c_1\,\boldsymbol{x}^{(1)}(t_0) + \cdots + c_n\,\boldsymbol{x}^{(n)}(t_0).$$

Introducing the notation

$$X(t) = \big[\boldsymbol{x}^{(1)}(t), \cdots, \boldsymbol{x}^{(n)(t)}\big], \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

the algebraic linear system has the form

$$\boldsymbol{x}(t_0) = X(t_0)\,\boldsymbol{c}.$$

This algebraic system has a unique solution $\boldsymbol{c}$ for every source $\boldsymbol{x}(t_0)$ iff the matrix $X(t_0)$ is invertible. This matrix is invertible iff $\det\big(X(t_0)\big) \neq 0$. The generalization of Abel's Theorem to systems, Theorem 5.1.11, says that $\det\big(X(t_0)\big) \neq 0$ iff the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a fundamental set of solutions to the differential equation. This establishes the Theorem. □

Example 5.1.13. Find the general solution to differential equation in Example 5.1.5 and then use this general solution to find the solution of the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \qquad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

Solution: From Example 5.1.5 we know that the general solution of the differential equation above can be written as

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

Before imposing the initial condition on this general solution, it is convenient to write this general solution using a matrix valued function, X, as follows

$$\boldsymbol{x}(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = X(t)\,\boldsymbol{c},$$

where we introduced the solution matrix and the constant vector, respectively,

$$X(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The initial condition fixes the vector $\boldsymbol{c}$, that is, its components c_1, c_2, as follows,

$$\boldsymbol{x}(0) = X(0)\,\boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{c} = \big[X(0)\big]^{-1}\boldsymbol{x}(0).$$

Since the solution matrix X at $t = 0$ has the form,

$$X(0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \big[X(0)\big]^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

introducing $\big[X(0)\big]^{-1}$ in the equation for $\boldsymbol{c}$ above we get

$$\boldsymbol{c} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} c_1 = -1, \\ c_2 = 3. \end{cases}$$

We conclude that the solution to the initial value problem above is given by

$$\boldsymbol{x}(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

◁

5.1.5. The Wronskian and Abel's Theorem. From the proof of Theorem 5.1.8 above we see that it is convenient to introduce the notion of solution matrix and Wronskian of a set of n solutions to an $n \times n$ linear differential system,

Definition 5.1.10.

(a) A ***solution matrix*** *of any set of vector functions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$, solutions to a differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$, is the $n \times n$ matrix valued function*

$$X(t) = \big[\boldsymbol{x}^{(1)}(t), \cdots, \boldsymbol{x}^{(n)}(t)\big]. \tag{5.1.15}$$

X is called a ***fundamental matrix*** *iff the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a fundamental set.*

(b) The ***Wronskian*** *of the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is the function $W(t) = \det\big(X(t)\big)$.*

Remark: A fundamental matrix provides a more compact way to write the general solution of a differential equation. The general solution in Eq. (5.1.14) can be rewritten as

$$\boldsymbol{x}_{\text{gen}}(t) = c_1\boldsymbol{x}^{(1)}(t) + \cdots + c_n\boldsymbol{x}^{(n)}(t) = \big[\boldsymbol{x}^{(1)}(t), \cdots, \boldsymbol{x}^{(n)}(t)\big]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = X(t)\,\boldsymbol{c}, \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

This is a more compact notation for the general solution,

$$\boldsymbol{x}_{\text{gen}}(t) = X(t)\,\boldsymbol{c}. \tag{5.1.16}$$

Remark: The definition of the Wronskian in Def 5.1.10 agrees with the Wronskian of solutions to second order linear scalar equations given in Def. 2.1.9, § 2.1. We can see this relation if we compute the first order reduction of a second order equation. So, the Wronskian of two solutions y_1, y_2 of the second order equation $y'' + a_1 y' + a_0 y = 0$, is

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Now compute the first order reduction of the differential equation above, as in Theorem 5.1.4,

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -a_0 x_1 - a_1 x_2. \end{aligned}$$

The solutions y_1, y_2 define two solutions of the 2×2 linear system,

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}, \qquad \boldsymbol{x}^{(2)} = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}.$$

The Wronskian for the scalar equation coincides with the Wronskian for the system, because

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = \det\big([\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}]\big) = W.$$

Example 5.1.14. Find two fundamental matrices for the linear homogeneous system in Example 5.1.11.

Solution: One fundamental matrix is simple to find, we use the solutions in Example 5.1.11,

$$X = \big[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\big] \quad \Rightarrow \quad X(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix}.$$

A second fundamental matrix can be obtained multiplying by any nonzero constant each solution above. For example, another fundamental matrix is

$$\tilde{X} = \big[2\boldsymbol{x}^{(1)}, 3\boldsymbol{x}^{(2)}\big] \quad \Rightarrow \quad \tilde{X}(t) = \begin{bmatrix} 2e^{-2t} & -3e^{4t} \\ 2e^{-2t} & 3e^{4t} \end{bmatrix}.$$

$\lhd$

Example 5.1.15. Compute the Wronskian of the vector valued functions given in Example 5.1.11, that is, $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$.

Solution: The Wronskian is the determinant of the solution matrix, with the vectors placed in any order. For example, we can choose the order $\big[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\big]$. If we choose the order $\big[\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(1)}\big]$, this second Wronskian is the negative of the first one. Choosing the first order for the solutions, we get

$$W(t) = \det\big([\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}]\big) = \begin{vmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{vmatrix} = e^{-2t}\, e^{4t} + e^{-2t}\, e^{4t}.$$

We conclude that $W(t) = 2e^{2t}$. $\lhd$

Example 5.1.16. Show that the set of functions $\left\{ \boldsymbol{x}^{(1)} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}, \ \boldsymbol{x}^{(2)} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} \right\}$ is linearly independent for all $t \in \mathbb{R}$.

Solution: We compute the determinant of the matrix $X(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$, that is,

$$w(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -2e^{2t} - 2e^{2t} \quad \Rightarrow \quad w(t) = -4e^{2t} \neq 0 \quad t \in \mathbb{R}.$$

$\lhd$

We now generalize Abel's Theorem 2.1.12 from a single equation to an $n \times n$ linear system.

Theorem 5.1.11 (Abel). *The Wronskian function $W = \det\big(X(t)\big)$ of a solution matrix $X = \big[\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\big]$ of the linear system $\boldsymbol{x}' = A(t)\boldsymbol{x}$, where A is an $n \times n$ continuous matrix valued function on a domain $I \subset \mathbb{R}$, satisfies the differential equation*

$$W'(t) = \operatorname{tr}\big[A(t)\big]\, W(t). \tag{5.1.17}$$

where $\operatorname{tr}(A)$ is the trace of A. Hence W is given by

$$W(t) = W(t_0)\, e^{\alpha(t)}, \qquad \alpha(t) = \int_{t_0}^{t} \operatorname{tr}\big(A(\tau)\big)\, d\tau.$$

where t_0 is any point in I.

Remarks:

(a) In the case of a constant matrix A, the equation above for the Wronskian reduces to

$$W(t) = W(t_0)\, e^{\operatorname{tr}(A)\,(t-t_0)},$$

(b) The Wronskian function vanishes at a single point iff it vanishes identically for all $t \in I$.

(c) A consequence of (b): n solutions to the system $\boldsymbol{x}' = A(t)\boldsymbol{x}$ are linearly independent at the initial time t_0 iff they are linearly independent for every time $t \in I$.

Proof of Theorem 5.1.11: The proof is based in an identity satisfied by the determinant of certain matrix valued functions. The proof of this identity is quite involved, so we do not provide it here. The identity is the following: Every $n \times n$, differentiable, invertible, matrix valued function Z, with values $Z(t)$ for $t \in \mathbb{R}$, satisfies the identity:

$$\frac{d}{dt}\det(Z) = \det(Z)\operatorname{tr}\Big(Z^{-1}\frac{d}{dt}Z\Big).$$

We use this identity with any fundamental matrix $X = \big[\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\big]$ of the linear homogeneous differential system $\boldsymbol{x}' = A\boldsymbol{x}$. Recalling that the Wronskian $w(t) = \det\big(X(t)\big)$, the identity above says,

$$W'(t) = W(t)\operatorname{tr}\big[X^{-1}(t)\, X'(t)\big].$$

We now compute the derivative of the fundamental matrix,

$$X' = \big[\boldsymbol{x}^{(1)\,\prime}, \cdots, \boldsymbol{x}^{(n)\,\prime}\big] = \big[A\boldsymbol{x}^{(1)}, \cdots, A\boldsymbol{x}^{(n)}\big] = AX,$$

where the equation on the far right comes from the definition of matrix multiplication. Replacing this equation in the Wronskian equation we get

$$W'(t) = W(t)\operatorname{tr}\big(X^{-1}AX\big) = W(t)\operatorname{tr}\big(X\,X^{-1}\,A\big) = W(t)\operatorname{tr}(A),$$

where in the second equation above we used a property of the trace of three matrices: $\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$. Therefore, we have seen that the Wronskian satisfies the equation

$$W'(t) = \operatorname{tr}\big[A(t)\big]\, W(t),$$

This is a linear differential equation of a single function $W : \mathbb{R} \to \mathbb{R}$. We integrate it using the integrating factor method from Section 1.2. The result is

$$W(t) = W(t_0)\, e^{\alpha(t)}, \qquad \alpha(t) = \int_{t_0}^{t} \operatorname{tr}\big[A(\tau)\big]\, d\tau.$$

This establishes the Theorem. □

Example 5.1.17. Show that the Wronskian of the fundamental matrix constructed with the solutions given in Example 5.1.3 satisfies Eq. (5.1.17) above.

Solution: In Example 5.1.5 we have shown that the vector valued functions $\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$ are solutions to the system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$. The matrix

$$X(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}$$

is a fundamental matrix of the system, since its Wronskian is non-zero,

$$W(t) = \begin{vmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{vmatrix} = 4e^{t} - e^{t} \quad \Rightarrow \quad W(t) = 3e^{t}.$$

We need to compute the right-hand side and the left-hand side of Eq. (5.1.17) and verify that they coincide. We start with the left-hand side,

$$W'(t) = 3e^t = W(t).$$

The right-hand side is

$$\operatorname{tr}(A)\,W(t) = (3-2)\,W(t) = W(t).$$

Therefore, we have shown that $W(t) = \operatorname{tr}(A)\,W(t)$. $\lhd$

5.1.6. Exercises.

5.1.1.- .

5.1.2.- .

5.2. Solution Formulas

We find an explicit formula for the solutions of linear systems of differential equations with constant coefficients. We first consider homogeneous equations and later on we generalize the solution formula to nonhomogeneous equations with nonconstant sources. Both solution formulas for linear *systems* are obtained generalizing the *integrating factor method* for linear *scalar* equations used in § 1.1, 1.2. In this section we use the exponential of a matrix, so the reader should read Chapter 8, in particular § 8.3, and § 8.4.

We also study the particular case when the coefficient matrix of a linear differential system is diagonalizable. In this case we show a well-known formula for the general solution of linear systems that involves the eigenvalues and eigenvectors of the coefficient matrix. To obtain this formula we transform the coupled system into an uncoupled system, we solve the uncoupled system, and we transform the solution back to the original variables. Later on we use this formula for the general solution to construct a fundamental matrix for the linear system. We then relate this fundamental matrix to the exponential formula for the solutions of a general linear system we found using the integrating factor method.

5.2.1. Homogeneous Systems.

We find an explicit formula for the solutions of first order homogeneous linear systems of differential equations with constant coefficients. This formula is found using the integrating factor method introduced in § 1.1 and 1.2.

Theorem 5.2.1 (Homogeneous Systems)**.** *If A is an $n \times n$ matrix, $t_0 \in \mathbb{R}$ is an arbitrary constant, and $\boldsymbol{x}_0$ is any constant n-vector, then the initial value problem for the unknown n-vector valued function $\boldsymbol{x}$ given by*

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

has a unique solution given by the formula

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\,\boldsymbol{x}_0. \tag{5.2.1}$$

Remark: See § 8.4 for the definitions of the exponential of a square matrix. In particular, recall the following properties of e^{At}, for a constant square matrix A and any $s, t \in \mathbb{R}$:

$$\frac{d}{dt}e^{At} = A\,e^{At} = e^{At}A, \qquad \bigl(e^{At}\bigr)^{-1} = e^{-At}, \qquad e^{As}e^{At} = e^{A(s+t)}.$$

Proof of Theorem 5.2.1: We generalize to linear systems the integrating factor method used in § 1.1 to solve linear scalar equations. Therefore, rewrite the equation as $\boldsymbol{x}' - A\,\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero n-vector, and then multiply the equation on the left by e^{-At},

$$e^{-At}\boldsymbol{x}' - e^{-At}A\,\boldsymbol{x} = \boldsymbol{0} \quad\Rightarrow\quad e^{-At}\boldsymbol{x}' - A\,e^{-At}\,\boldsymbol{x} = \boldsymbol{0},$$

since $e^{-At}A = A\,e^{-At}$. We now use the properties of the matrix exponential to rewrite the system as

$$e^{-At}\boldsymbol{x}' + \bigl(e^{-At}\bigr)'\,\boldsymbol{x} = \boldsymbol{0} \quad\Rightarrow\quad \bigl(e^{-At}\boldsymbol{x}\bigr)' = \boldsymbol{0}.$$

If we integrate in the last equation above, and we denote by $\boldsymbol{c}$ a constant n-vector, we get

$$e^{-At}\boldsymbol{x}(t) = \boldsymbol{c} \quad\Rightarrow\quad \boldsymbol{x}(t) = e^{At}\boldsymbol{c},$$

where we used $\bigl(e^{-At}\bigr)^{-1} = e^{At}$. If we now evaluate at $t = t_0$ we get the constant vector $\boldsymbol{c}$,

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0) = e^{At_0}\boldsymbol{c} \quad\Rightarrow\quad \boldsymbol{c} = e^{-At_0}\boldsymbol{x}_0.$$

Using this expression for $\boldsymbol{c}$ in the solution formula above we get

$$\boldsymbol{x}(t) = e^{At}e^{-At_0}\boldsymbol{x}_0 \quad\Rightarrow\quad \boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0.$$

This establishes the Theorem. □

Example 5.2.1. Compute the exponential function e^{At} and use it to express the vector-valued function $\boldsymbol{x}$ solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

Solution: The exponential of a matrix is simple to compute in the case that the matrix is diagonalizable. So we start checking whether matrix A above is diagonalizable. Theorem 8.3.8 says that a 2×2 matrix is diagonalizable if it has two eigenvectors not proportional to each other. In oder to find the eigenvectors of A we need to compute its eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} (1-\lambda) & 2 \\ 2 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 4.$$

The roots of the characteristic polynomial are

$$(\lambda - 1)^2 = 4 \quad \Leftrightarrow \quad \lambda_{\pm} = 1 \pm 2 \quad \Leftrightarrow \quad \lambda_{+} = 3, \quad \lambda_{-} = -1.$$

The eigenvectors corresponding to the eigenvalue $\lambda_+ = 3$ are the solutions $\boldsymbol{v}^+$ of the linear system $(A - 3I_2)\boldsymbol{v}^+ = \boldsymbol{0}$. To find them, we perform Gauss operations on the matrix

$$A - 3I_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1^+ = v_2^+ \quad \Rightarrow \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue $\lambda_- = -1$ are the solutions $\boldsymbol{v}^-$ of the linear system $(A + I_2)\boldsymbol{v}^- = \boldsymbol{0}$. To find them, we perform Gauss operations on the matrix

$$A + I_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1^- = -v_2^- \quad \Rightarrow \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Summarizing, the eigenvalues and eigenvectors of matrix A are following,

$$\lambda_+ = 3, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -1, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then, Theorem 8.3.8 says that the matrix A is diagonalizable, that is $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Now Theorem **??** says that the exponential of At is given by

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so we conclude that

$$e^{At} = \frac{1}{2}\begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix}. \tag{5.2.2}$$

Finally, we get the solution to the initial value problem above,

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 = \frac{1}{2}\begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

In components, this means

$$\boldsymbol{x}(t) = \frac{1}{2}\begin{bmatrix} (x_{01} + x_{02})\, e^{3t} + (x_{01} - x_{02})\, e^{-t} \\ (x_{01} + x_{02})\, e^{3t} - (x_{01} - x_{02})\, e^{-t} \end{bmatrix}.$$

◁

5.2.2. Homogeneous Diagonalizable Systems. A linear system $\boldsymbol{x}' = A\,\boldsymbol{x}$ is diagonalizable iff the coefficient matrix A is diagonalizable, which means that there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (See § 8.3 for a review on diagonalizable matrices.) *The solution formula in Eq. (5.2.1) includes diagonalizable systems.* But when a system is diagonalizable there is a simpler way to solve it. One transforms the system, where all the equations are coupled together, into a decoupled system. One can solve the decoupled system, one equation at a time. The last step is to transform the solution back to the original variables. We show how this idea works in a very simple example.

Example 5.2.2. Find functions x_1, x_2 solutions of the first order, 2×2, constant coefficients, homogeneous differential system

$$\begin{aligned} x_1' &= x_1 - x_2,\\ x_2' &= -x_1 + x_2. \end{aligned}$$

Solution: As it is usually the case, the equations in the system above are coupled. One must know the function x_2 in order to integrate the first equation to obtain the function x_1. Similarly, one has to know function x_1 to integrate the second equation to get function x_2. The system is coupled; one cannot integrate one equation at a time. One must integrate the whole system together.

However, the coefficient matrix of the system above is diagonalizable. In this case the equations can be decoupled. If we add the two equations equations, and if we subtract the second equation from the first, we obtain, respectively,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

To see more clearly what we have done, let us introduce the new unknowns $y_1 = x_1 + x_2$, and $y_2 = x_1 - x_2$, and rewrite the equations above with these new unknowns,

$$y_1' = 0, \qquad y_2' = 2y_2.$$

We have decoupled the original system. The equations for x_1 and x_2 are coupled, but we have found a linear combination of the equations such that the equations for y_1 and y_2 are not coupled. We now solve each equation independently of the other.

$$\begin{aligned} y_1' = 0 \quad &\Rightarrow \quad y_1 = c_1,\\ y_2' = 2y_2 \quad &\Rightarrow \quad y_2 = c_2 e^{2t}, \end{aligned}$$

with c_1, $c_2 \in \mathbb{R}$. Having obtained the solutions for the decoupled system, we now transform back the solutions to the original unknown functions. From the definitions of y_1 and y_2 we see that

$$x_1 = \frac{1}{2}\,(y_1 + y_2), \qquad x_2 = \frac{1}{2}\,(y_1 - y_2).$$

We conclude that for all c_1, $c_2 \in \mathbb{R}$ the functions x_1, x_2 below are solutions of the 2×2 differential system in the example, namely,

$$x_1(t) = \frac{1}{2}\,(c_1 + c_2\,e^{2t}), \qquad x_2(t) = \frac{1}{2}\,(c_1 - c_2\,e^{2t}).$$

◁

The equations for x_1 and x_2 in the example above are coupled, so we found an appropriate linear combination of the equations and the unknowns such that the equations for the new unknown functions, y_1 and y_2, are decoupled. We integrated each equation independently of the other, and we finally transformed the solutions back to the original unknowns

x_1 and x_2. The key step is to find the transformation from x_1, x_2 to y_1, y_2. For general systems this transformation may not exist. It exists, however, for diagonalizable systems.

Remark: Recall Theorem 8.3.8, which says that an $n \times n$ matrix A diagonalizable iff A has a linearly independent set of n eigenvectors. Furthermore, if λ_i, $\boldsymbol{v}^{(i)}$ are eigenpairs of A, then the decomposition $A = PDP^{-1}$ holds for

$$P = \big[\boldsymbol{v}^{(i)}, \cdots, \boldsymbol{v}^{(n)}\big], \qquad D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big].$$

For diagonalizable systems of homogeneous differential equations there is a formula for the general solution that includes the eigenvalues and eigenvectors of the coefficient matrix.

Theorem 5.2.2 (Homogeneous Diagonalizable Systems)**.** *If the $n \times n$ constant matrix A is diagonalizable, with a set of linearly independent eigenvectors $\big\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the system $\boldsymbol{x}' = A\,\boldsymbol{x}$ has a general solution*

$$\boldsymbol{x}_{\text{gen}}(t) = c_1\, e^{\lambda_1 t}\, \boldsymbol{v}^{(1)} + \cdots + c_n\, e^{\lambda_n t}\, \boldsymbol{v}^{(n)}. \tag{5.2.3}$$

Furthermore, every initial value problem $\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$, with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, has a unique solution for every initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$, where the constants $c_1, \cdots, c_n$ are solution of the algebraic linear system

$$\boldsymbol{x}_0 = c_1\, e^{\lambda_1 t_0}\, \boldsymbol{v}^{(1)} + \cdots + c_n\, e^{\lambda_1 t_0}\, \boldsymbol{v}^{(n)}. \tag{5.2.4}$$

Remark: We show two proofs of this Theorem. The first one is just a verification that the expression in Eq. (5.2.3) satisfies the differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$. The second proof follows the same idea presented to solve Example 5.2.2. We decouple the system, we solve the uncoupled system, and we transform back to the original unknowns. The differential system is decoupled when written in the basis of eigenvectors of the coefficient matrix.

First proof of Theorem 5.2.2: Each function $\boldsymbol{x}^{(i)} = e^{\lambda_i t}\, \boldsymbol{v}^{(i)}$, for $i = 1, \cdots, n$, is solution of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$, because

$$\boldsymbol{x}^{(i)\prime} = \lambda_i\, e^{\lambda_i t}\, \boldsymbol{v}^{(i)}, \qquad A\boldsymbol{x}^{(i)} = A\left(e^{\lambda_i t}\, \boldsymbol{v}^{(i)}\right) = e^{\lambda_i t}\, A\, \boldsymbol{v}^{(i)} = \lambda_i\, e^{\lambda_i t}\, \boldsymbol{v}^{(i)},$$

hence $\boldsymbol{x}^{(i)\prime} = A\,\boldsymbol{x}^{(i)}$. Since A is diagonalizable, the set

$$\big\{\boldsymbol{x}^{(1)}(t) = e^{\lambda_1 t}\, \boldsymbol{v}^{(1)}, \cdots, \boldsymbol{x}^{(n)}(t) = e^{\lambda_n t}\, \boldsymbol{v}^{(n)}\big\}$$

is a fundamental set of solutions to the system. Therefore, the superposition property says that the general solution to the system is

$$\boldsymbol{x}(t) = c_1\, e^{\lambda_1 t}\, \boldsymbol{v}^{(1)} + \cdots + c_n\, e^{\lambda_n t}\, \boldsymbol{v}^{(n)}.$$

The constants $c_1, \cdots, c_n$ are computed by evaluating the equation above at t_0 and recalling the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. This establishes the Theorem. □

Remark: In the proof above we verify that the functions $\boldsymbol{x}^{(i)} = e^{\lambda_i t}\, \boldsymbol{v}^{(i)}$ are solutions, but we do not say why we choose these functions in the first place. In the proof below we construct the solutions, and we find that they are the ones given in the proof above.

Second proof of Theorem 5.2.2: Since the coefficient matrix A is diagonalizable, there exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Introduce this expression into the differential equation and multiplying the whole equation by P^{-1},

$$P^{-1}\boldsymbol{x}'(t) = P^{-1}\big(PDP^{-1}\big)\, \boldsymbol{x}(t).$$

Notice that to multiply the differential system by the matrix P^{-1} means to perform a very particular type of linear combinations among the equations in the system. This is the linear combination that *decouples* the system. Indeed, since matrix A is constant, so is P and D. In particular $P^{-1}\boldsymbol{x}' = \big(P^{-1}\boldsymbol{x}\big)'$, hence

$$(P^{-1}\boldsymbol{x})' = D\,\big(P^{-1}\,\boldsymbol{x}\big).$$

Define the new variable $\boldsymbol{y} = \big(P^{-1}\boldsymbol{x}\big)$. The differential equation is now given by

$$\boldsymbol{y}'(t) = D\,\boldsymbol{y}(t).$$

Since matrix D is diagonal, the system above is a *decoupled* for the variable $\boldsymbol{y}$. Transform the initial condition too, that is, $P^{-1}\boldsymbol{x}(t_0) = P^{-1}\boldsymbol{x}_0$, and use the notation $\boldsymbol{y}_0 = P^{-1}\boldsymbol{x}_0$, so we get the initial condition in terms of the $\boldsymbol{y}$ variable,

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0.$$

Solve the decoupled initial value problem $\boldsymbol{y}'(t) = D\,\boldsymbol{y}(t)$,

$$\left.\begin{matrix} y_1'(t) = \lambda_1\,y_1(t), \\ \vdots \\ y_n'(t) = \lambda_n\,y_n(t), \end{matrix}\right\} \quad\Rightarrow\quad \left\{\begin{matrix} y_1(t) = c_1\,e^{\lambda_1 t}, \\ \vdots \\ y_n(t) = c_n\,e^{\lambda_n t}, \end{matrix}\right\} \quad\Rightarrow\quad \boldsymbol{y}(t) = \begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix}.$$

Once $\boldsymbol{y}$ is found, we transform back to $\boldsymbol{x}$,

$$\boldsymbol{x}(t) = P\,\boldsymbol{y}(t) = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big] \begin{bmatrix} c_1\,e^{\lambda_1 t} \\ \vdots \\ c_n\,e^{\lambda_n t} \end{bmatrix} = c_1\,e^{\lambda_1 t}\,\boldsymbol{v}^{(1)} + \cdots + c_n\,e^{\lambda_n t}\,\boldsymbol{v}^{(n)}.$$

This is Eq. (5.2.3). Evaluating it at t_0 we get Eq. (5.2.4). This establishes the Theorem. $\square$

Example 5.2.3. Find the vector-valued function $\boldsymbol{x}$ solution to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution: First we need to find out whether the coefficient matrix A is diagonalizable or not. Theorem 8.3.8 says that a 2×2 matrix is diagonalizable iff there exists a linearly independent set of two eigenvectors. So we start computing the matrix eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} (1-\lambda) & 2 \\ 2 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 4.$$

The roots of the characteristic polynomial are

$$(\lambda - 1)^2 = 4 \quad\Leftrightarrow\quad \lambda_{\pm} = 1 \pm 2 \quad\Leftrightarrow\quad \lambda_+ = 3, \quad \lambda_- = -1.$$

The eigenvectors corresponding to the eigenvalue $\lambda_+ = 3$ are the solutions $\boldsymbol{v}^+$ of the linear system $(A - 3I_2)\boldsymbol{v}^+ = \boldsymbol{0}$. To find them, we perform Gauss operations on the matrix

$$A - 3I_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad\Rightarrow\quad v_1^+ = v_2^+ \quad\Rightarrow\quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue $\lambda_- = -1$ are the solutions $\boldsymbol{v}^-$ of the linear system $(A + I_2)\boldsymbol{v}^- = \boldsymbol{0}$. To find them, we perform Gauss operations on the matrix

$$A + I_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad\Rightarrow\quad v_1^- = -v_2^- \quad\Rightarrow\quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Summarizing, the eigenvalues and eigenvectors of matrix A are following,

$$\lambda_+ = 3, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -1, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Once we have the eigenvalues and eigenvectors of the coefficient matrix, Eq. (5.2.3) gives us the general solution

$$\boldsymbol{x}(t) = c_+ \, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_- \, e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where the coefficients c_+ and c_- are solutions of the initial condition equation

$$c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We conclude that $c_+ = 5/2$ and $c_- = -1/2$, hence

$$\boldsymbol{x}(t) = \frac{5}{2} \, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \, e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix}.$$

◁

Example 5.2.4. Find the general solution to the 2×2 differential system

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: We need to find the eigenvalues and eigenvectors of the coefficient matrix A. But they were found in Example 8.3.4, and the result is

$$\lambda_+ = 4, \quad \boldsymbol{v}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda_- = -2, \quad \boldsymbol{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

With these eigenpairs we construct fundamental solutions of the differential equation,

$$\begin{aligned} \lambda_+ &= 4, & \boldsymbol{v}^{(+)} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} & &\Rightarrow & \boldsymbol{x}^{(+)}(t) &= e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \lambda_- &= -2, & \boldsymbol{v}^{(-)} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} & &\Rightarrow & \boldsymbol{x}^{(-)}(t) &= e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore, the general solution of the differential equation is

$$\boldsymbol{x}(t) = c_+ \, e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_- \, e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad c_+, c_- \in \mathbb{R}.$$

◁

The formula in Eq. 5.2.3 is a remarkably simple way to write the general solution of the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ in the case A is diagonalizable. It is a formula easy to remember, you just add all terms of the form $e^{\lambda_i t}\, \boldsymbol{v}^i$, where λ_i, $\boldsymbol{v}^i$ is any eigenpair of A. But this formula is not the best one to write down solutions to initial value problems. As you can see in Theorem 5.2.2, we did not provide a formula for that. We only said that the constants $c_1, \cdots, c_n$ are the solutions of the algebraic linear system in (5.2.4). But we did not write the solution for the c's. It is too complicated in this notation, though it is not difficult to do it on every particular case, as we did near the end of Example 5.2.3.

A simple way to introduce the initial condition in the expression of the solution is with a fundamental matrix, which we introduced in Eq. (5.1.10).

Theorem 5.2.3 (Fundamental Matrix Expression)**.** *If the $n \times n$ constant matrix A is diagonalizable, with a set of linearly independent eigenvectors $\big\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then, the initial value problem $\boldsymbol{x}' = A\,\boldsymbol{x}$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ has a unique solution given by*

$$\boldsymbol{x}(t) = X(t)X(t_0)^{-1}\,\boldsymbol{x}_0 \tag{5.2.5}$$

where $X(t) = \big[e^{\lambda_1 t}\,\boldsymbol{v}^{(1)}, \cdots, e^{\lambda_n t}\,\boldsymbol{v}^{(n)}\big]$ is a fundamental matrix of the system.

Proof of Theorem 5.2.3: If we choose fundamental solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$ to be

$$\big\{\boldsymbol{x}^{(1)}(t) = e^{\lambda_1 t}\,\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{x}^{(n)}(t) = e^{\lambda_n t}\,\boldsymbol{v}^{(n)}\big\},$$

then the associated fundamental matrix is

$$X(t) = \big[e^{\lambda_1 t}\,\boldsymbol{v}^{(1)}, \cdots, e^{\lambda_n t}\,\boldsymbol{v}^{(n)}\big],$$

We use this fundamental matrix to write the general solution of the differential system as

$$\mathbf{x}_{\text{gen}}(t) = c_1\,e^{\lambda_1 t}\,\mathbf{v}^{(1)} + \cdots + c_n\,e^{\lambda_n t}\,\mathbf{v}^{(n)} = X(t)\,\boldsymbol{c}, \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

The equation from the initial condition is now

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0) = X(t_0)\,\boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{c} = X(t_0)^{-1}\boldsymbol{x}_0,$$

which makes sense, since $X(t)$ is an invertible matrix for all t where it is defined. Using this formula for the constant vector $\boldsymbol{c}$ we get,

$$\boldsymbol{x}(t) = X(t)X(t_0)^{-1}\,\boldsymbol{x}_0.$$

This establishes the Theorem. $\square$

Example 5.2.5. Find a fundamental matrix for the system below and use it to write down the general solution to the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution: One way to find a fundamental matrix of a system is to start computing the eigenvalues and eigenvectors of the coefficient matrix. The differential equation in this Example is the same as the one given in Example 5.2.3, where we found that the eigenvalues and eigenvectors of the coefficient matrix are

$$\lambda_+ = 3, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -1, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We see that the coefficient matrix is diagonalizable, so with the eigenpairs above we can construct a fundamental set of solutions,

$$\Big\{\boldsymbol{x}^{(+)}(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{x}^{(-)}(t) = e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\Big\}.$$

From here we construct a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}.$$

Then we have the general solution $\boldsymbol{x}_{\text{gen}}(t) = X(t)\,\boldsymbol{c}$, where $\boldsymbol{c} = \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$, that is,

$$\boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}_{\text{gen}}(t) = c_+\, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_+\, e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

◁

Example 5.2.6. Use the fundamental matrix found in Example 5.2.5 to write down the solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution: In Example 5.2.5 we found the general solution to the differential equation,

$$\boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

The initial condition has the form

$$\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \boldsymbol{x}(0) = X(0)\,\boldsymbol{c} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

We need to compute the inverse of matrix $X(0)$,

$$X(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so we compute the constant vector $\boldsymbol{c}$,

$$\begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

So the solution to the initial value problem is,

$$\boldsymbol{x}(t) = X(t)X(0)^{-1}\boldsymbol{x}_0 \quad \Leftrightarrow \quad \boldsymbol{x}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

If we compute the matrix on the last equation, explicitly, we get,

$$\boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

◁

Remark: In the Example 5.2.6 above we found that, for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, holds

$$X(t)X(0)^{-1} = \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix},$$

which is precisely the same as the expression for e^{At} we found in Eq. (5.2.2) in Example 5.2.2,

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix}.$$

This is not a coincidence. If a matrix A is diagonalizable, then $e^{A(t-t_0)} = X(t)X(t_0)^{-1}$. We summarize this result in the theorem below.

Theorem 5.2.4 (Exponential for Diagonalizable Systems)**.** *If an $n\times n$ matrix A has linearly independent eigenvectors $\{\boldsymbol{v}^{(1)},\cdots,\boldsymbol{v}^{(n)}\}$ with corresponding eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$, then*

$$e^{A(t-t_0)} = X(t)X(t_0)^{-1},$$

where $X(t) = \big[e^{\lambda_1 t}\,\boldsymbol{v}^{(1)},\cdots,e^{\lambda_n t}\,\boldsymbol{v}^{(n)}\big]$.

Proof of Theorem 5.2.4: We start rewriting the formula for the fundamental matrix given in Theorem 5.2.3,

$$X(t) = \big[\boldsymbol{v}^{(1)}e^{\lambda_1 t},\cdots,\boldsymbol{v}^{(n)}e^{\lambda_n t}\big] = \big[\boldsymbol{v}^{(1)},\cdots,\boldsymbol{v}^{(n)}\big]\begin{bmatrix} e^{\lambda_1 t} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda_n t}\end{bmatrix},$$

The diagonal matrix on the last equation above can be written as

$$\begin{bmatrix} e^{\lambda_1 t} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda_n t}\end{bmatrix} = \operatorname{diag}\big[e^{\lambda_1 t},\cdots,e^{\lambda_n t}\big].$$

If we recall the exponential of a matrix defined in § 8.4, we can see that the matrix above is an exponential, since

$$\operatorname{diag}\big[e^{\lambda_1 t},\cdots,e^{\lambda_n t}\big] = e^{Dt}, \qquad \text{where} \qquad Dt = \operatorname{diag}\big[\lambda_1 t,\cdots,\lambda_n t\big].$$

One more thing, let us denote $P = \big[\boldsymbol{v}^{(1)},\cdots,\boldsymbol{v}^{(n)}\big]$, as we did in § 8.3. If we use these two expressions into the formula for X above, we get

$$X(t) = P\,e^{Dt}.$$

Using properties of invertible matrices, given in § 8.2, and the properties of the exponential of a matrix, given in § 8.4, we get

$$X(t_0)^{-1} = \big(Pe^{Dt_0}\big)^{-1} = e^{-Dt_0}P^{-1},$$

where we used that $\big(e^{Dt_0}\big)^{-1} = e^{-Dt_0}$. These manipulations lead us to the formula

$$X(t)X(t_0)^{-1} = Pe^{Dt}e^{-Dt_0}P^{-1} \quad\Leftrightarrow\quad X(t)X(t_0)^{-1} = Pe^{D(t-t_0)}P^{-1}.$$

Since A is diagonalizable, with $A = PDP^{-1}$, we known from § 8.4 that

$$Pe^{D(t-t_0)}P^{-1} = e^{A(t-t_0)}.$$

We conclude that

$$X(t)X(t_0)^{-1} = e^{A(t-t_0)}.$$

This establishes the Theorem. □

Example 5.2.7. Verify Theorem 5.2.4 for matrix $A = \begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}$ and $t_0 = 0$.

Solution: We known from Example 5.2.4 that the eigenpairs of matrix A above are

$$\lambda_+ = 4, \quad \boldsymbol{v}^{(+)} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{and} \qquad \lambda_- = -2, \quad \boldsymbol{v}^{(-)} = \begin{bmatrix}-1\\1\end{bmatrix}.$$

This means that a fundamental matrix for A is

$$X(t) = \begin{bmatrix} e^{4t} & -e^{-2t}\\ e^{4t} & e^{-2t}\end{bmatrix}.$$

This fundamental matrix at $t = 0$ is

$$X(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Rightarrow \quad X(0)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Therefore we get that

$$X(t)X(0)^{-1} = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

On the other hand, e^{At} can be computed using the formula $e^{At} = P e^{Dt} P^{-1}$, where

$$Dt = \begin{bmatrix} 4t & 0 \\ 0 & -2t \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \quad \Rightarrow \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then we get

$$e^{At} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so we get

$$e^{At} = \frac{1}{2}\begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

We conclude that $e^{At} = X(t)X(0)^{-1}$. ◁

5.2.3. Nonhomogeneous Systems.

The solution formula of an initial value problem for an nonhomogeneous linear system is a generalization of the solution formula for a scalar equation given in § 1.2. We use the *integrating factor method*, just as in § 1.2.

Theorem 5.2.5 (Nonhomogeneous Systems)**.** *If A is a constant $n \times n$ matrix and $\boldsymbol{b}$ is a continuous n-vector function, then the initial value problem*

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t) + \boldsymbol{b}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

has a unique solution for every initial condition $t_0 \in \mathbb{R}$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ given by

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + e^{A(t-t_0)} \int_{t_0}^{t} e^{-A(\tau - t_0)}\,\boldsymbol{b}(\tau)\,d\tau. \tag{5.2.6}$$

Remark: Since $e^{\pm At_0}$ are constant matrices, an equivalent expression for Eq. (5.2.6) is

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + e^{At} \int_{t_0}^{t} e^{-A\tau}\,\boldsymbol{b}(\tau)\,d\tau.$$

In the case of an homogeneous system, $\boldsymbol{b} = \boldsymbol{0}$, we get Eq. (5.2.1).

Proof of Theorem 5.2.5: We generalize to linear systems the integrating factor method used in § 1.2 to solve linear scalar equations. Therefore, rewrite the equation as $\boldsymbol{x}' - A\,\boldsymbol{x} = \boldsymbol{b}$, and then multiply the equation on the left by e^{-At},

$$e^{-At}\boldsymbol{x}' - e^{-At}A\,\boldsymbol{x} = e^{-At}\,\boldsymbol{b} \quad \Rightarrow \quad e^{-At}\boldsymbol{x}' - A\,e^{-At}\,\boldsymbol{x} = e^{-At}\,\boldsymbol{b},$$

since $e^{-At}A = A\,e^{-At}$. We now use the formulas for the derivative of an exponential,

$$e^{-At}\boldsymbol{x}' + \left(e^{-At}\right)'\boldsymbol{x} = e^{-At}\,\boldsymbol{b} \quad \Rightarrow \quad \left(e^{-At}\boldsymbol{x}\right)' = e^{-At}\,\boldsymbol{b}.$$

If we integrate on the interval $[t_0, t]$ the last equation above, we get

$$e^{-At}\boldsymbol{x}(t) - e^{-At_0}\boldsymbol{x}(t_0) = \int_{t_0}^{t} e^{-A\tau}\,\boldsymbol{b}(\tau)\,d\tau.$$

If we reorder terms and we use that $\left(e^{-At}\right)^{-1} = e^{At}$,

$$\boldsymbol{x}(t) = e^{At} e^{-At_0} \boldsymbol{x}_0 + e^{At} \int_{t_0}^{t} e^{-A\tau}\, \boldsymbol{b}(\tau)\, d\tau.$$

Finally, using the group property of the exponential, $e^{At}\, e^{-At_0} = e^{A(t-t_0)}$, we get

$$\boldsymbol{x}(t) = e^{A(t-t_0)} \boldsymbol{x}_0 + e^{At} \int_{t_0}^{t} e^{-A\tau}\, \boldsymbol{b}(\tau)\, d\tau,$$

This establishes the Theorem. □

Example 5.2.8. Find the vector-valued solution $\boldsymbol{x}$ to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x} + \boldsymbol{b}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution: In Example 5.2.3 we have found the eigenvalues and eigenvectors of the coefficient matrix, and the result is

$$\lambda_1 = 3, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_2 = -1, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors above say that A is diagonalizable,

$$A = PDP^{-1}, \qquad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

We also know how to compute the exponential of a diagonalizable matrix,

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so we conclude that

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \quad \Rightarrow \quad e^{-At} = \frac{1}{2} \begin{bmatrix} (e^{-3t} + e^{t}) & (e^{-3t} - e^{t}) \\ (e^{-3t} - e^{t}) & (e^{-3t} + e^{t}) \end{bmatrix}.$$

The solution to the initial value problem above is,

$$\boldsymbol{x}(t) = e^{At} \boldsymbol{x}_0 + e^{At} \int_0^t e^{-A\tau}\, \boldsymbol{b}\, d\tau.$$

Since

$$e^{At} \boldsymbol{x}_0 = \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix},$$

in a similar way

$$e^{-A\tau} \boldsymbol{b} = \frac{1}{2} \begin{bmatrix} (e^{-3\tau} + e^{\tau}) & (e^{-3\tau} - e^{\tau}) \\ (e^{-3\tau} - e^{\tau}) & (e^{-3\tau} + e^{\tau}) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3e^{-3\tau} - e^{\tau} \\ 3e^{-3\tau} + e^{\tau} \end{bmatrix}.$$

Integrating the last expresion above, we get

$$\int_0^t e^{-A\tau}\, \boldsymbol{b}\, d\tau = \frac{1}{2} \begin{bmatrix} -e^{-3t} - e^{t} \\ -e^{-3t} + e^{t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, we get

$$\boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \left[\frac{1}{2} \begin{bmatrix} -e^{-3t} - e^{t} \\ -e^{-3t} + e^{t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right].$$

Multiplying the matrix-vector product on the second term of the left-hand side above,

$$\boldsymbol{x}(t) = \frac{1}{2}\begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (e^{3t} + e^{-t}) \\ (e^{3t} - e^{-t}) \end{bmatrix}.$$

We conclude that the solution to the initial value problem above is

$$\boldsymbol{x}(t) = \begin{bmatrix} 3e^{3t} + e^{-t} - 1 \\ 3e^{3t} - e^{-t} \end{bmatrix}.$$

◁

Remark: The formula in Eq. (5.2.6) is also called the *variation of parameters formula*. The reason is that Eq. (5.2.6) can be seen as

$$\boldsymbol{x}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)$$

where $\boldsymbol{x}_h(t) = e^{A(t-t_0)}$ is solution of the homogeneous equation $\boldsymbol{x}' = A\,\boldsymbol{x}$, and $\boldsymbol{x}_p$ is a particular solution of the nonhomogeneous equation. One can generalize the variation of parameters method to get $\boldsymbol{x}_p$ as follows,

$$\boldsymbol{x}_p(t) = X(t)\,\boldsymbol{u}(t),$$

where $X(t)$ is a fundamental matrix of the homogeneous system, and $\boldsymbol{u}$ are functions to be determined. If one introduce this $\boldsymbol{x}_p$ in the nonhomogeneous equation, one gets

$$X'\,\boldsymbol{u} + X\,\boldsymbol{u}' = AX\,\boldsymbol{u} + \boldsymbol{b}$$

One can prove that the fundamental matrix satisfies the differential equation $X' = AX$. If we use this equation for X in the equation above, we get

$$AX\,\boldsymbol{u} + X\,\boldsymbol{u}' = AX\,\boldsymbol{u} + \boldsymbol{b} \quad\Rightarrow\quad X\boldsymbol{u}' = \boldsymbol{b}$$

so we get the equation

$$\boldsymbol{u}' = X^{-1}\,\boldsymbol{b} \quad\Rightarrow\quad \boldsymbol{u}(t) = \int_{t_0}^{t} \big[X(\tau)\big]^{-1}\,\boldsymbol{b}(\tau)\,d\tau.$$

Therefore, a particular solution found with this method is

$$\boldsymbol{x}_p(t) = X(t)\int_{t_0}^{t} X(\tau)^{-1}\,\boldsymbol{b}(\tau)\,d\tau.$$

If we use that $X(t_0)^{-1}$ and $X(t_0)$ are constant matrices, we get

$$\begin{aligned} \boldsymbol{x}_p(t) &= X(t)\,\big[X(t_0)^{-1}X(t_0)\big]\int_{t_0}^{t} X(\tau)^{-1}\,\boldsymbol{b}(\tau)\,d\tau \\ &= X(t)X(t_0)^{-1}\int_{t_0}^{t} X(t_0)X(\tau)^{-1}\,\boldsymbol{b}(\tau)\,d\tau \\ &= X(t)X(t_0)^{-1}\int_{t_0}^{t} \big[X(\tau)X(t_0)^{-1}\big]^{-1}\,\boldsymbol{b}(\tau)\,d\tau. \end{aligned}$$

Now, one can also prove that $e^{A(t-t_0)} = X(t)X(t_0)^{-1}$ for all $n \times n$ coefficient matrices, no just diagonalizable matrices. If we use that formula we get

$$\boldsymbol{x}_p(t) = e^{A(t-t_0)}\int_{t_0}^{t} e^{-A(\tau-t_0)}\,\boldsymbol{b}(\tau)\,d\tau.$$

So we recover the expression in Eq. (5.2.6) for $\boldsymbol{x} = \boldsymbol{x}_h + \boldsymbol{x}_p$. This is why Eq. (5.2.6) is also called the variation of parameters formula.

5.2.4. Exercises.

5.2.1.- Use the exponential formula in Eq. (5.2.1) to find the solution of the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

where

$$A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}.$$

5.2.2.- Use the exponential formula in Eq. (5.2.1) to find the solution of the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

where

$$A = \begin{bmatrix} 5 & -4 \\ -8 & -7 \end{bmatrix}.$$

5.2.3.- * Follow the proof of Theorem 5.2.2 to find the general solution of the system

$$\boldsymbol{x}' = A\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

where

$$A = \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of the coefficient matrix.

(b) Find functions y_1, y_2 of the form

$$y_1 = \alpha_{11}x_1 + \alpha_{12}x_2$$
$$y_2 = \alpha_{21}x_1 + \alpha_{22}x_2,$$

so that the differential equation for

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is decoupled.

(c) If we write the differential equation for $\boldsymbol{y}$ as

$$\boldsymbol{y}' = B\,\boldsymbol{y},$$

find the matrix B.

(d) Solve the differential equation for $\boldsymbol{y}$.

(e) Use the solution $\boldsymbol{y}$ to find the solution $\boldsymbol{x}$ of the original differential equation. Write the solution as

$$\boldsymbol{x}(t) = c_1\,\boldsymbol{x}^{(1)}(t) + c_2\,\boldsymbol{x}^{(2)}(t),$$

and give explicit expressions for $\boldsymbol{x}^{(1)}(t)$ and $\boldsymbol{x}^{(2)}(t)$.

5.3. Two-Dimensional Homogeneous Systems

2×2 linear systems are important not only by themselves but as approximations of more complicated nonlinear systems. They are important by themselves because 2×2 systems are simple enough so their solutions can be computed and classified. But they are non-trivial enough so their solutions describe several situations including exponential decays and oscillations. In this Section we study 2×2 systems in detail and we classify them according the eigenvalues of the coefficient matrix. In a later Chapter we will use them as approximations of more complicated systems.

5.3.1. Diagonalizable Systems.

Consider a 2×2 constant coefficient, homogeneous linear differential system,

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where we assume that all matrix coefficents are real constants. The characteristic polynomial of the coefficient matrix is $p(\lambda) = \det(A - \lambda I)$. This is a polynomial degree two with real coefficients. Hence it may have two distinct roots–real or complex–or one repeated real root. In the case that the roots are distinct the coefficient matrix is diagonalizable, see Chapter 8. In the case that the root is repeated, the coefficient matrix may or may not be diagonalizable. Theorem 5.2.2 holds for a diagonalizable 2×2 coefficient matrix and state it below in the notation we use for 2×2 systems.

Theorem 5.3.1 (Diagonalizable Systems)**.** *If the 2×2 constant matrix A is diagonalizable with eigenpairs $\lambda_\pm$, $\boldsymbol{v}^{(\pm)}$, then the general solution of $\boldsymbol{x}' = A\,\boldsymbol{x}$ is*

$$\boldsymbol{x}_{\text{gen}}(t) = c_+\, e^{\lambda_+ t}\, \boldsymbol{v}^{(+)} + c_-\, e^{\lambda_- t}\, \boldsymbol{v}^{(-)}. \tag{5.3.1}$$

We classify the 2×2 linear systems by the eigenvalues of their coefficient matrix:

(i) The eigenvalues λ_+, λ_- are real and distinct;
(ii) The eigenvalues $\lambda_\pm = \alpha \pm \beta i$ are distinct and complex, with $\lambda_+ = \overline{\lambda_-}$;
(iii) The eigenvalues $\lambda_+ = \lambda_- = \lambda_0$ is repeated and real.

We now provide a few examples of systems on each of the cases above, starting with an example of case *(i)*.

Example 5.3.1. Find the general solution of the 2×2 linear system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: We have computed in Example 8.3.4 the eigenpairs of the coefficient matrix,

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This coefficient matrix has distinct real eigenvalues, so the general solution to the differential equation is

$$\boldsymbol{x}_{\text{gen}}(t) = c_+\, e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_-\, e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

◁

We now focus on case *(ii)*. The coefficient matrix is real-valued with the complex-valued eigenvalues. In this case each eigenvalue is the complex conjugate of the other. A similar result is true for $n \times n$ real-valued matrices. When such $n \times n$ matrix has a complex eigenvalue λ, there is another eigenvalue $\overline{\lambda}$. A similar result holds for the respective eigenvectors.

Theorem 5.3.2 (Conjugate Pairs)**.** *If an $n \times n$ real-valued matrix A has a complex eigenpair λ, $\boldsymbol{v}$, then the complex conjugate pair $\overline{\lambda}$, $\overline{\boldsymbol{v}}$ is also an eigenpair of matrix A.*

Proof of Theorem 5.3.2: Complex conjugate the eigenvalue eigenvector equation for λ and $\boldsymbol{v}$, and recall that matrix A is real-valued, hence $\overline{A} = A$. We obtain,

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \quad \Leftrightarrow \quad A\overline{\boldsymbol{v}} = \overline{\lambda}\, \overline{\boldsymbol{v}},$$

This establishes the Theorem. □

Complex eigenvalues of a matrix with real coefficients are always complex conjugate pairs. Same it's true for their respective eigenvectors. So they can be written in terms of their real and imaginary parts as follows,

$$\lambda_{\pm} = \alpha \pm i\beta, \qquad \boldsymbol{v}^{(\pm)} = \boldsymbol{a} \pm i\boldsymbol{b}, \tag{5.3.2}$$

where α, $\beta \in \mathbb{R}$ and $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n$.

The general solution formula in Eq. (5.3.1) still holds in the case that A has complex eigenvalues and eigenvectors. The main drawback of this formula is similar to what we found in Chapter 2. It is difficult to separate real-valued from complex-valued solutions. The fix to that problem is also similar to the one found in Chapter 2: Find a real-valued fundamental set of solutions. The following result holds for $n \times n$ systems.

Theorem 5.3.3 (Complex and Real Solutions)**.** *If $\lambda_{\pm} = \alpha \pm i\beta$ are eigenvalues of an $n \times n$ constant matrix A with eigenvectors $\boldsymbol{v}^{(\pm)} = \boldsymbol{a} \pm i\boldsymbol{b}$, where α, $\beta \in \mathbb{R}$ and $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n$, and $n \geqslant 2$, then a linearly independent set of two complex-valued solutions to $\boldsymbol{x}' = A\,\boldsymbol{x}$ is*

$$\big\{ \boldsymbol{x}^{(+)}(t) = e^{\lambda_+ t}\, \boldsymbol{v}^{(+)},\ \boldsymbol{x}^{(-)}(t) = e^{\lambda_- t}\, \boldsymbol{v}^{(-)},\ \big\}. \tag{5.3.3}$$

Furthermore, a linearly independent set of two real-valued solutions to $\boldsymbol{x}' = A\,\boldsymbol{x}$ is given by

$$\big\{ \boldsymbol{x}^{(1)}(t) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\, e^{\alpha t},\ \boldsymbol{x}^{(2)}(t) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\, e^{\alpha t} \big\}. \tag{5.3.4}$$

Proof of Theorem 5.3.3: Theorem 8.3.9 implies the set in (5.3.3) is a linearly independent set. The new information in Theorem 5.3.3 above is the real-valued solutions in Eq. (5.3.4). They are obtained from Eq. (5.3.3) as follows:

$$\begin{aligned}
\boldsymbol{x}^{(\pm)} &= (\boldsymbol{a} \pm i\boldsymbol{b})\, e^{(\alpha \pm i\beta)t} \\
&= e^{\alpha t} (\boldsymbol{a} \pm i\boldsymbol{b})\, e^{\pm i\beta t} \\
&= e^{\alpha t} (\boldsymbol{a} \pm i\boldsymbol{b})\, \big(\cos(\beta t) \pm i \sin(\beta t)\big) \\
&= e^{\alpha t} \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big) \pm i e^{\alpha t} \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big).
\end{aligned}$$

Since the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$ is linear, the functions below are also solutions,

$$\begin{aligned}
\boldsymbol{x}^{(1)} &= \frac{1}{2}\big(\boldsymbol{x}^{+} + \boldsymbol{x}^{-}\big) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\, e^{\alpha t}, \\
\boldsymbol{x}^{(2)} &= \frac{1}{2i}\big(\boldsymbol{x}^{+} - \boldsymbol{x}^{-}\big) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\, e^{\alpha t}.
\end{aligned}$$

This establishes the Theorem. □

Example 5.3.2. Find a real-valued set of fundamental solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}. \tag{5.3.5}$$

Solution: Fist find the eigenvalues of matrix A above,

$$0 = \begin{vmatrix} (2-\lambda) & 3 \\ -3 & (2-\lambda) \end{vmatrix} = (\lambda-2)^2 + 9 \quad \Rightarrow \quad \lambda_{\pm} = 2 \pm 3i.$$

Then find the respective eigenvectors. The one corresponding to λ_+ is the solution of the homogeneous linear system with coefficients given by

$$\begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

Therefore the eigenvector $\boldsymbol{v}^{+} = \begin{bmatrix} v^{+}_1 \\ v^{+}_2 \end{bmatrix}$ is given by

$$v^{+}_1 = -i v^{+}_2 \quad \Rightarrow \quad v^{+}_2 = 1, \quad v^{+}_1 = -i, \quad \Rightarrow \quad \boldsymbol{v}^{+} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_{+} = 2 + 3i.$$

The second eigenvector is the complex conjugate of the eigenvector found above, that is,

$$\boldsymbol{v}^{-} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda_{-} = 2 - 3i.$$

Notice that

$$\boldsymbol{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i.$$

Then, the real and imaginary parts of the eigenvalues and of the eigenvectors are given by

$$\alpha = 2, \qquad \beta = 3, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

So a real-valued expression for a fundamental set of solutions is given by

$$\boldsymbol{x}^{1} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right) e^{2t} \quad \Rightarrow \quad \boldsymbol{x}^{1} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\boldsymbol{x}^{2} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right) e^{2t} \quad \Rightarrow \quad \boldsymbol{x}^{2} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

◁

We end with case (iii). There are no many possibilities left for a 2×2 real matrix that both is diagonalizable and has a repeated eigenvalue. Such matrix must be proportional to the identity matrix.

Theorem 5.3.4. *Every 2×2 diagonalizable matrix with repeated eigenvalue λ_0 has the form*

$$A = \lambda_0 I.$$

Proof of Theorem 5.3.4: Since matrix A diagonalizable, there exists a matrix P invertible such that $A = PDP^{-1}$. Since A is 2×2 with a repeated eigenvalue λ_0, then

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda_0 \, I_2.$$

Put these two fatcs together,

$$A = P\lambda_0 I P^{-1} = \lambda_0 P \, P^{-1} = \lambda_0 I.$$

□

Remark: The general solution $\boldsymbol{x}_{\text{gen}}$ for $\boldsymbol{x}' = \lambda I\, \boldsymbol{x}$ is simple to write. Since any non-zero 2-vector is an eigenvector of $\lambda_0 I_2$, we choos the linearly independent set

$$\Big\{ \boldsymbol{v}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{v}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big\}.$$

Using these eigenvectors we can write the general solution,

$$\boldsymbol{x}_{\text{gen}}(t) = c_1\, e^{\lambda_0 t}\, \boldsymbol{v}^{(1)} + c_2\, e^{\lambda_0 t}\, \boldsymbol{v}^{(2)} = c_1\, e^{\lambda_0 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\, e^{\lambda_0 t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}_{\text{gen}}(t) = e^{\lambda t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

5.3.2. Non-Diagonalizable Systems.

A 2×2 linear systems might not be diagonalizable. This can happen only when the coefficient matrix has a repeated eigenvalue and all eigenvectors are proportional to each other. If we denote by λ the repeated eigenvalue of a 2×2 matrix A, and by $\boldsymbol{v}$ an associated eigenvector, then one solution to the differential system $\boldsymbol{x}' = A\, \boldsymbol{x}$ is

$$\boldsymbol{x}^{(1)}(t) = e^{\lambda t}\, \boldsymbol{v}.$$

Every other eigenvector $\tilde{\boldsymbol{v}}$ associated with λ is proportional to $\boldsymbol{v}$. So any solution of the form $\tilde{\boldsymbol{v}}\, e^{\lambda t}$ is proportional to the solution above. The next result provides a linearly independent set of two solutions to the system $\boldsymbol{x}' = A\boldsymbol{x}$ associated with the repeated eigenvalue λ.

Theorem 5.3.5 (Repeated Eigenvalue)**.** *If an 2×2 matrix A has a repeated eigenvalue λ with only one associated eigen-direction, given by the eigenvector $\boldsymbol{v}$, then the differential system $\boldsymbol{x}'(t) = A\, \boldsymbol{x}(t)$ has a linearly independent set of solutions*

$$\big\{ \boldsymbol{x}^1(t) = e^{\lambda t}\, \boldsymbol{v}, \quad \boldsymbol{x}^2(t) = e^{\lambda t}\, \big(\boldsymbol{v}\, t + \boldsymbol{w}\big) \big\},$$

where the vector $\boldsymbol{w}$ is one of infinitely many solutions of the algebraic linear system

$$(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}. \tag{5.3.6}$$

Remark: The eigenvalue λ is the precise number that makes matrix $(A - \lambda I)$ not invertible, that is, $\det(A - \lambda I) = 0$. This implies that an algebraic linear system with coefficient matrix $(A - \lambda I)$ is not consistent for every source. Nevertheless, the Theorem above says that Eq. (5.3.6) has solutions. The fact that the source vector in that equation is $\boldsymbol{v}$, an eigenvector of A, is crucial to show that this system is consistent.

Proof of Theorem 5.3.5: One solution to the differential system is $\boldsymbol{x}^{(1)}(t) = e^{\lambda t}\, \boldsymbol{v}$. Inspired by the reduction order method we look for a second solution of the form

$$\boldsymbol{x}^{(2)}(t) = e^{\lambda t}\, \boldsymbol{u}(t).$$

Inserting this function into the differential equation $\boldsymbol{x}' = A\, \boldsymbol{x}$ we get

$$\boldsymbol{u}' + \lambda\, \boldsymbol{u} = A\, \boldsymbol{u} \quad \Rightarrow \quad (A - \lambda I)\, \boldsymbol{u} = \boldsymbol{u}'.$$

We now introduce a power series expansion of the vector-valued function $\boldsymbol{u}$,

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \boldsymbol{u}_1 t + \boldsymbol{u}_2 t^2 + \cdots,$$

into the differential equation above,

$$(A - \lambda I)(\boldsymbol{u}_0 + \boldsymbol{u}_1 t + \boldsymbol{u}_2 t^2 + \cdots) = (\boldsymbol{u}_1 + 2\boldsymbol{u}_2 t + \cdots).$$

If we evaluate the equation above at $t = 0$, and then its derivative at $t = 0$, and so on, we get the following infinite set of linear algebraic equations

$$\begin{aligned}(A - \lambda I)\boldsymbol{u}_0 &= \boldsymbol{u}_1,\\ (A - \lambda I)\boldsymbol{u}_1 &= 2\boldsymbol{u}_2,\\ (A - \lambda I)\boldsymbol{u}_2 &= 3\boldsymbol{u}_3\\ &\vdots\end{aligned}$$

Here is where we use Cayley-Hamilton's Theorem. Recall that the characteristic polynomial $p(\tilde{\lambda}) = \det(A - \tilde{\lambda} I)$ has the form

$$p(\tilde{\lambda}) = \tilde{\lambda}^2 - \operatorname{tr}(A)\,\tilde{\lambda} + \det(A).$$

Cayley-Hamilton Theorem says that the matrix-valued polynomial $p(A) = 0$, that is,

$$A^2 - \operatorname{tr}(A)\,A + \det(A)\,I = 0.$$

Since in the case we are interested in matrix A has a repeated root λ, then

$$p(\tilde{\lambda}) = (\tilde{\lambda} - \lambda)^2 = \tilde{\lambda}^2 - 2\lambda\,\tilde{\lambda} + \lambda^2.$$

Therefore, Cayley-Hamilton Theorem for the matrix in this Theorem has the form

$$0 = A^2 - 2\lambda\,A + \lambda^2\,I \quad \Rightarrow \quad (A - \lambda I)^2 = 0.$$

This last equation is the one we need to solve the system for the vector-valued $\boldsymbol{u}$. Multiply the first equation in the system by $(A - \lambda I)$ and use that $(A - \lambda I)^2 = 0$, then we get

$$\boldsymbol{0} = (A - \lambda I)^2\boldsymbol{u}_0 = (A - \lambda I)\,\boldsymbol{u}_1 \quad \Rightarrow \quad (A - \lambda I)\boldsymbol{u}_1 = \boldsymbol{0}.$$

This implies that $\boldsymbol{u}_1$ is an eigenvector of A with eigenvalue λ. We can denote it as $\boldsymbol{u}_1 = \boldsymbol{v}$. Using this information in the rest of the system we get

$$\begin{aligned}(A - \lambda I)\boldsymbol{u}_0 &= \boldsymbol{v},\\ (A - \lambda I)\boldsymbol{v} &= 2\boldsymbol{u}_2 \quad \Rightarrow \quad \boldsymbol{u}_2 = \boldsymbol{0},\\ (A - \lambda I)\boldsymbol{u}_2 &= 3\boldsymbol{u}_3 \quad \Rightarrow \quad \boldsymbol{u}_3 = \boldsymbol{0},\\ &\vdots\end{aligned}$$

We conclude that all terms $\boldsymbol{u}_2 = \boldsymbol{u}_3 = \cdots = \boldsymbol{0}$. Denoting $\boldsymbol{u}_0 = \boldsymbol{w}$ we obtain the following system of algebraic equations,

$$\begin{aligned}(A - \lambda I)\boldsymbol{w} &= \boldsymbol{v},\\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}.\end{aligned}$$

For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ solution of the system above we get $\boldsymbol{u}(t) = \boldsymbol{w} + t\boldsymbol{v}$. This means that the second solution to the differential equation is

$$\boldsymbol{x}^{(2)}(t) = e^{\lambda t}\,(t\boldsymbol{v} + \boldsymbol{w}).$$

This establishes the Theorem. □

Example 5.3.3. Find the fundamental solutions of the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \frac{1}{4}\begin{bmatrix}-6 & 4\\ -1 & -2\end{bmatrix}.$$

Solution: As usual, we start finding the eigenvalues and eigenvectors of matrix A. The former are the solutions of the characteristic equation

$$0 = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix} = \Big(\lambda + \frac{3}{2}\Big)\Big(\lambda + \frac{1}{2}\Big) + \frac{1}{4} = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2.$$

Therefore, there solution is the repeated eigenvalue $\lambda = -1$. The associated eigenvectors are the vectors $\boldsymbol{v}$ solution to the linear system $(A + I)\boldsymbol{v} = \boldsymbol{0}$,

$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 2v_2.$$

Choosing $v_2 = 1$, then $v_1 = 2$, and we obtain

$$\lambda = -1, \qquad \boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Any other eigenvector associated to $\lambda = -1$ is proportional to the eigenvector above. The matrix A above is not diagonalizable. So. we follow Theorem 5.3.5 and we solve for a vector $\boldsymbol{w}$ the linear system $(A + I)\boldsymbol{w} = \boldsymbol{v}$. The augmented matrix for this system is given by,

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 1 & -2 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad w_1 = 2w_2 - 4.$$

We have obtained infinitely many solutions given by

$$\boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

As one could have imagined, given any solution $\boldsymbol{w}$, the $c\boldsymbol{v} + \boldsymbol{w}$ is also a solution for any $c \in \mathbb{R}$. We choose the simplest solution given by

$$\boldsymbol{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

Therefore, a fundamental set of solutions to the differential equation above is formed by

$$\boldsymbol{x}^{(1)}(t) = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}^{(2)}(t) = e^{-t} \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right). \tag{5.3.7}$$

◁

5.3.3. Exercises.

5.3.1.- .

5.3.2.- .

5.4. Two-Dimensional Phase Portraits

Figures are easier to understand than words. Words are easier to understand than equations. The qualitative behavior of a function is often simpler to visualize from a graph than from an explicit or implicit expression of the function.

Take, for example, the differential equation

$$y'(t) = \sin(y(t)).$$

This equation is separable and the solution can be obtained using the techniques in Section 1.3. They lead to the following implicit expression for the solution y,

$$-\ln\big|\csc(y) + \cot(y)\big| = t + c.$$

Although this is an exact expression for the solution of the differential equation, the qualitative behavior of the solution is not so simple to understand from this formula. The graph of the solution, however, given on the right, provides us with a clear picture of the solution behavior. In this particular case the graph of the solution can be computed from the equation itself, without the need to solve the equation.

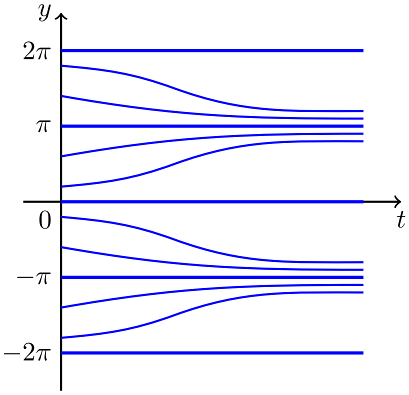


FIGURE 1. Several solutions of the equation $y' = \sin(y)$

In the case of 2×2 systems the solution vector has the form

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Two functions define the solution vector. In this case one usually graphs each component of the solution vector, x_1 and x_2, as functions of t. There is, however, another way to graph a 2-vector-valued function: plot the whole vector $\boldsymbol{x}(t)$ at t on the plane x_1, x_2. Each vector $\boldsymbol{x}(t)$ is represented by its end point, while the whole solution $\boldsymbol{x}$ represents a line with arrows pointing in the direction of increasing t. Such a figure is called a *phase diagram* or *phase portrait*.

In the case that the solution vector $\boldsymbol{x}(t)$ is interpreted as the position function of a particle moving in a plane at the time t, the curve given in the phase portrait is the trajectory of the particle. The arrows added to this trajectory indicate the motion of the particle as time increases.

In this Section we say how to plot phase portraits. We focus on solutions to the systems studied in the previous Section 5.3–2×2 homogeneous, constant coefficient linear systems

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t). \tag{5.4.1}$$

Theorem 5.3.1 spells out the general solution in the case the coefficient matrix is diagonalizable with eigenpairs $\lambda_\pm$, $\boldsymbol{v}^\pm$. The general solution is given by

$$\boldsymbol{x}_{\text{gen}}(t) = c_+ \boldsymbol{v}^+ e^{\lambda_+ t} + c_- \boldsymbol{v}^- e^{\lambda_- t}. \tag{5.4.2}$$

Solutions with real distinct eigenvalues are essentially different from solutions with complex eigenvalues. Those differences can be seen in their phase portraits. Both solution types are essentially different from solutions with a repeated eigenvalue. We now study each case.

5.4.1. Real Distinct Eigenvalues. We study the system in (5.4.1) in the case that matrix A has two real eigenvalues $\lambda_+ \neq \lambda_-$. The case where one eigenvalues vanishes is left one of the exercises at the end of the Section. We study the case where both eigenvalues are non-zero. Two non-zero eigenvalues belong to one of hte following cases:

(i) $\lambda_+ > \lambda_- > 0$, both eigenvalues positive;
(ii) $\lambda_+ > 0 > \lambda_-$, one eigenvalue negative and the other positive;
(iii) $0 > \lambda_+ > \lambda_-$, both eigenvalues negative.

In a phase portrait the solution vector $\boldsymbol{x}(t)$ at t is displayed on the plane x_1, x_2. The whole vector is shown, only the end point of the vector is shown for $t \in (-\infty, \infty)$. The result is a curve in the x_1, x_2 plane. One usually adds arrows to determine the direction of increasing t. A phase portrait contains several curves, each one corresponding to a solution given in Eq. (5.4.2) for particular choice of constants c_+ and c_-. A phase diagram can be sketched by following these few steps:

(a) Plot the eigenvectors $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ corresponding to the eigenvalues λ_+ and λ_-.
(b) Draw the whole lines parallel to these vectors and passing through the origin. These straight lines correspond to solutions with either c_+ or c_- zero.
(c) Draw arrows on these lines to indicate how the solution changes as the variable t increases. If t is interpreted as time, the arrows indicate how the solution changes into the future. The arrows point towards the origin if the corresponding eigenvalue λ is negative, and they point away form the origin if the eigenvalue is positive.
(d) Find the non-straight curves correspond to solutions with both coefficient c_+ and c_- non-zero. Again, arrows on these curves indicate the how the solution moves into the future.

Case $\lambda_+ > \lambda_- > 0$.

Example 5.4.1. Sketch the phase diagram of the solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \frac{1}{4}\begin{bmatrix} 11 & 3 \\ 1 & 9 \end{bmatrix}. \tag{5.4.3}$$

Solution: The characteristic equation for this matrix A is given by

$$\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 3, \\ \lambda_- = 2. \end{cases}$$

One can show that the corresponding eigenvectors are given by

$$\boldsymbol{v}^+ = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}^- = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

So the general solution to the differential equation above is given by

$$\boldsymbol{x}(t) = c_+\,\boldsymbol{v}^+ e^{\lambda_+ t} + c_-\,\boldsymbol{v}^- e^{\lambda_- t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = c_+ \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{3t} + c_- \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{2t}.$$

In Fig. 2 we have sketched four curves, each representing a solution $\boldsymbol{x}$ corresponding to a particular choice of the constants c_+ and c_-. These curves actually represent eight different solutions, for eight different choices of the constants c_+ and c_-, as is described below. The arrows on these curves represent the change in the solution as the variable t grows. Since both eigenvalues are positive, the length of the solution vector always increases as t increases. The straight lines correspond to the following four solutions:

$$c_+ = 1,\ c_- = 0; \quad c_+ = 0,\ c_- = 1; \quad c_+ = -1,\ c_- = 0; \quad c_+ = 0,\ c_- = -1.$$

The curved lines on each quadrant correspond to the following four solutions:

$$c_+ = 1,\ c_- = 1; \quad c_+ = 1,\ c_- = -1; \quad c_+ = -1,\ c_- = 1; \quad c_+ = -1,\ c_- = -1.$$

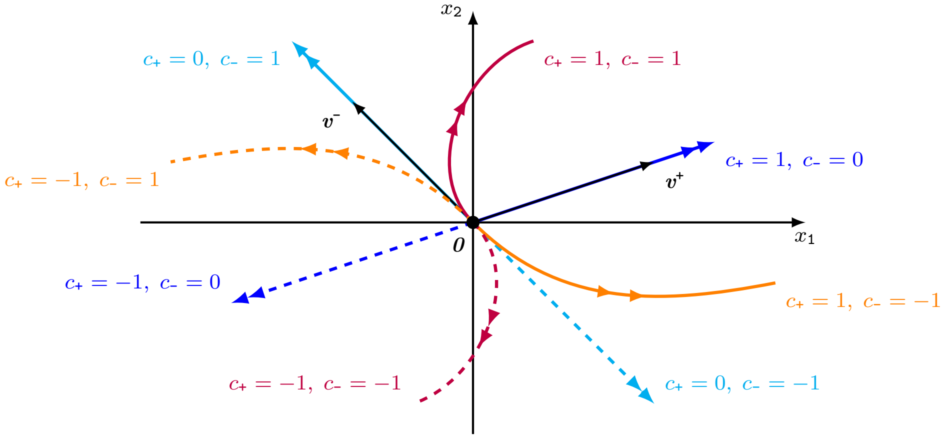


FIGURE 2. Eight solutions to Eq. (5.4.3), where $\lambda_+ > \lambda_- > 0$. The trivial solution $\boldsymbol{x} = \boldsymbol{0}$ is called an **unstable point**.

$\triangleleft$

Case $\lambda_+ > 0 > \lambda_-$.

Example 5.4.2. Sketch the phase diagram of the solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}. \tag{5.4.4}$$

Solution: In Example **??** we computed the eigenvalues and eigenvectors of the coefficient matrix, and the result was

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

In that Example we also computed the general solution to the differential equation above,

$$\boldsymbol{x}(t) = c_+\, \boldsymbol{v}^+ e^{\lambda_+ t} + c_-\, \boldsymbol{v}^- e^{\lambda_- t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

In Fig. 3 we have sketched four curves, each representing a solution $\boldsymbol{x}$ corresponding to a particular choice of the constants c_+ and c_-. These curves actually represent eight different solutions, for eight different choices of the constants c_+ and c_-, as is described below. The arrows on these curves represent the change in the solution as the variable t grows. The part of the solution with positive eigenvalue increases exponentially when t grows, while the part of the solution with negative eigenvalue decreases exponentially when t grows. The straight lines correspond to the following four solutions:

$$c_+ = 1,\ c_- = 0; \quad c_+ = 0,\ c_- = 1; \quad c_+ = -1,\ c_- = 0; \quad c_+ = 0,\ c_- = -1.$$

The curved lines on each quadrant correspond to the following four solutions:

$$c_+ = 1,\ c_- = 1; \quad c_+ = 1,\ c_- = -1; \quad c_+ = -1,\ c_- = 1; \quad c_+ = -1,\ c_- = -1.$$

$\triangleleft$

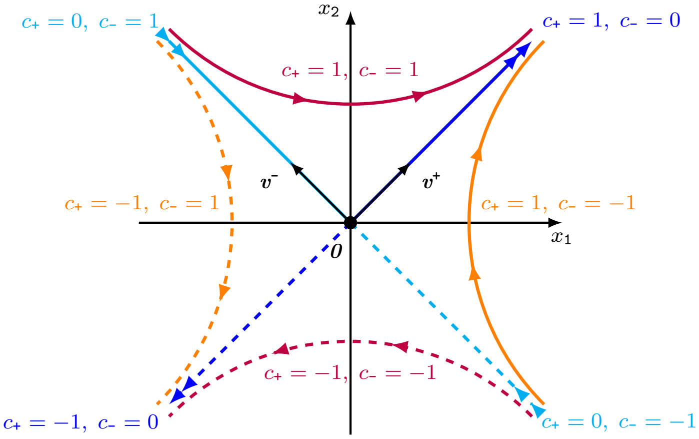


FIGURE 3. Several solutions to Eq. (5.4.4), $\lambda_+ > 0 > \lambda_-$. The trivial solution $\boldsymbol{x} = \boldsymbol{0}$ is called a **saddle point**.

Case $0 > \lambda_+ > \lambda_-$**.**

Example 5.4.3. Sketch the phase diagram of the solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \frac{1}{4}\begin{bmatrix} -9 & 3 \\ 1 & -11 \end{bmatrix}. \tag{5.4.5}$$

Solution: The characteristic equation for this matrix A is given by

$$\det(A - \lambda I) = \lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_+ = -2, \\ \lambda_- = -3. \end{cases}$$

One can show that the corresponding eigenvectors are given by

$$\boldsymbol{v}^+ = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}^- = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

So the general solution to the differential equation above is given by

$$\boldsymbol{x}(t) = c_+\, \boldsymbol{v}^+ e^{\lambda_+ t} + c_-\, \boldsymbol{v}^- e^{\lambda_- t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = c_+ \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-2t} + c_- \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-3t}.$$

In Fig. 4 we have sketched four curves, each representing a solution $\boldsymbol{x}$ corresponding to a particular choice of the constants c_+ and c_-. These curves actually represent eight different solutions, for eight different choices of the constants c_+ and c_-, as is described below. The arrows on these curves represent the change in the solution as the variable t grows. Since both eigenvalues are negative, the length of the solution vector always decreases as t grows and the solution vector always approaches zero. The straight lines correspond to the following four solutions:

$$c_+ = 1,\ c_- = 0; \quad c_+ = 0,\ c_- = 1; \quad c_+ = -1,\ c_- = 0; \quad c_+ = 0,\ c_- = -1.$$

The curved lines on each quadrant correspond to the following four solutions:

$$c_+ = 1,\ c_- = 1; \quad c_+ = 1,\ c_- = -1; \quad c_+ = -1,\ c_- = 1; \quad c_+ = -1,\ c_- = -1.$$

◁

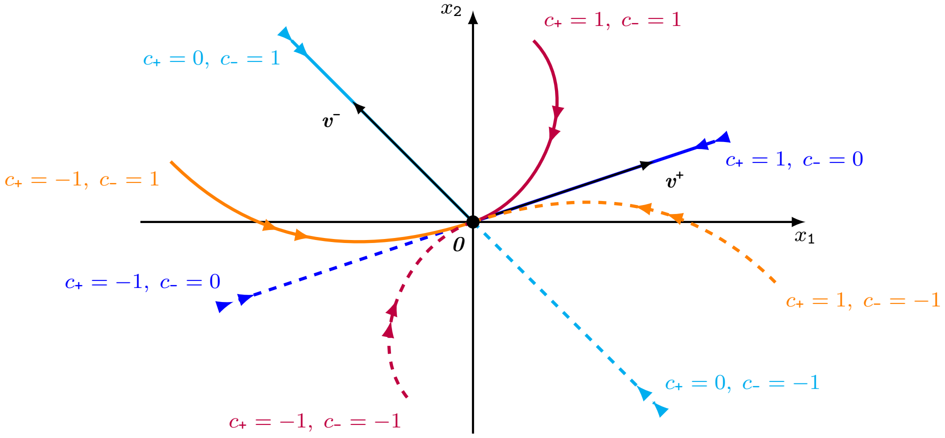


FIGURE 4. Several solutions to Eq. (5.4.5), where $0 > \lambda_+ > \lambda_-$. The trivial solution $\boldsymbol{x} = \boldsymbol{0}$ is called a **stable point**.

5.4.2. Complex Eigenvalues.

A real-valued matrix may have complex-valued eigenvalues. These complex eigenvalues come in pairs, because the matrix is real-valued. If λ is one of these complex eigenvalues, then $\overline{\lambda}$ is also an eigenvalue. A usual notation is $\lambda_\pm = \alpha \pm i\beta$, with $\alpha,\ \beta \in \mathbb{R}$. The same happens with their eigenvectors, which are written as $\boldsymbol{v}^\pm = \boldsymbol{a} \pm i\boldsymbol{b}$, with $\boldsymbol{a},\ \boldsymbol{b} \in \mathbb{R}^n$, in the case of an $n \times n$ matrix. When the matrix is the coefficient matrix of a differential equation,

$$\boldsymbol{x}' = A\,\boldsymbol{x},$$

the solutions $\boldsymbol{x}^+(t) = \boldsymbol{v}^+ e^{\lambda_+ t}$ and $\boldsymbol{x}^-(t) = \boldsymbol{v}^- e^{\lambda_- t}$ are complex-valued. In the previous Section we presented Theorem 5.3.3, which provided real-valued solutions for the differential equation. They are the real part and the imaginary part of the solution $\boldsymbol{x}^+$, given by

$$\boldsymbol{x}^1(t) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\,e^{\alpha t}, \qquad \boldsymbol{x}^2(t) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\,e^{\alpha t}. \tag{5.4.6}$$

These real-valued solutions are used to draw phase portraits. We start with an example.

Example 5.4.4. Find a real-valued set of fundamental solutions to the differential equation below and sketch a phase portrait, where

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: We have found in Example 5.3.2 that the eigenvalues and eigenvectors of the coefficient matrix are

$$\lambda_\pm = 2 \pm 3i, \quad \boldsymbol{v}^\pm = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}.$$

Writing them in real and imaginary parts, $\lambda_\pm = \alpha \pm i\beta$ and $\boldsymbol{v}^\pm = \boldsymbol{a} \pm i\boldsymbol{b}$, we get

$$\alpha = 2, \qquad \beta = 3, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

These eigenvalues and eigenvectors imply the following real-valued fundamental solutions,

$$\Big\{\boldsymbol{x}^1(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},\ \boldsymbol{x}^2(t) = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}\Big\}. \tag{5.4.7}$$

The phase diagram of these two fundamental solutions is given in Fig. 5 below. There is also a circle given in that diagram, corresponding to the trajectory of the vectors

$$\tilde{\boldsymbol{x}}^{1}(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} \qquad \tilde{\boldsymbol{x}}^{2}(t) = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix}.$$

The phase portrait of these functions is a circle, since they are unit vector-valued functions–they have length one. ◁

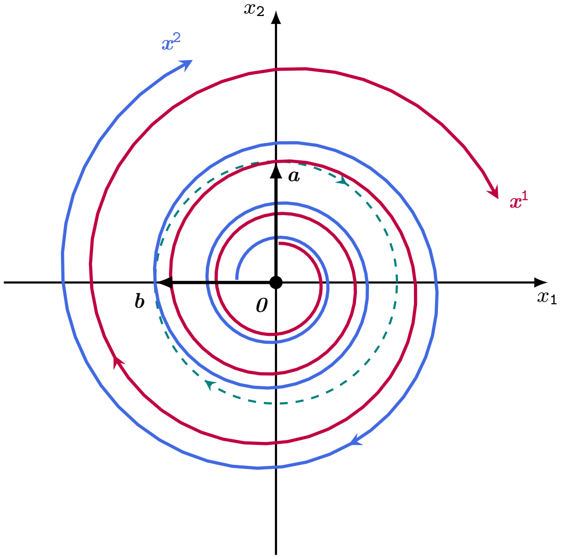


FIGURE 5. The graph of the fundamental solutions $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ in Eq. (5.4.7).

Suppose that the coefficient matris of a 2×2 differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ has complex eigenvalues and eigenvectors

$$\lambda_{\pm} = \alpha \pm i\beta, \qquad \boldsymbol{v}^{\pm} = \boldsymbol{a} \pm i\,\boldsymbol{b}.$$

We have said that real-valued fundamental solutions are given by

$$\boldsymbol{x}^{1}(t) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\,e^{\alpha t}, \qquad \boldsymbol{x}^{2}(t) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\,e^{\alpha t}.$$

We now sketch phase portraits of these solutions for a few choices of α, $\boldsymbol{a}$ and $\boldsymbol{b}$. We start fixing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and plotting phase diagrams for solutions having $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The result can be seen in Fig. 6. For $\alpha > 0$ the solutions spiral outward as t increases, and for $\alpha < 0$ the solutions spiral inwards to the origin as t increases. The rotation direction is from vector $\boldsymbol{b}$ towards vector $\boldsymbol{a}$. The solution vector $\boldsymbol{0}$, is called unstable for $\alpha > 0$ and stable for $\alpha < 0$.

We now change the direction of vector $\boldsymbol{b}$, and we repeat the three phase portraits given above; for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The result is given in Fig. 7. Comparing Figs. 6 and 7 shows that the relative directions of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ determines the rotation direction of the solutions as t increases.

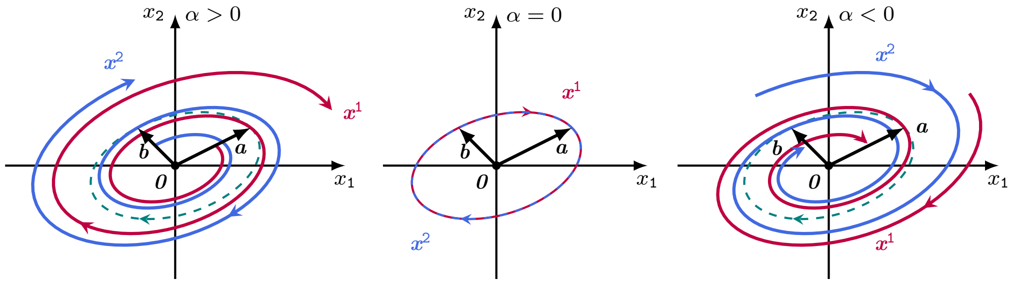


FIGURE 6. Fundamental solutions $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in Eq. (5.4.6) for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The relative positions of $\boldsymbol{a}$ and $\boldsymbol{b}$ determines the rotation direction. Compare with Fig. 7.

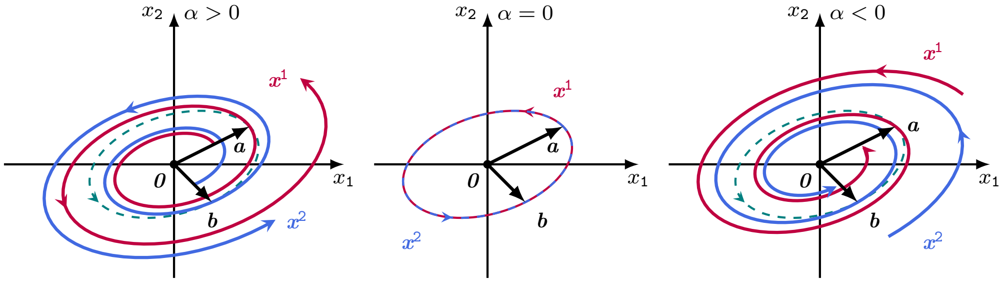


FIGURE 7. Fundamental solutions $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in Eq. (5.4.6) for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The relative positions of $\boldsymbol{a}$ and $\boldsymbol{b}$ determines the rotation direction. Compare with Fig. 6.

5.4.3. Repeated Eigenvalues. A matrix with repeated eigenvalues may or may not be diagonalizable. If a 2×2 matrix A is diagonalizable with repeated eigenvalues, then by Theorem 5.3.4 this matrix is proportional to the identity matrix, $A = \lambda_0 I$, with λ_0 the repeated eigenvalue. We saw in Section 5.3 that the general solution of a differential system with such coefficient matrix is

$$\boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda_0 t}.$$

Phase portraits of these solutions are just straight lines, starting from the origin for $\lambda_0 > 0$, or ending at the origin for $\lambda_0 < 0$.

Non-diagonalizable 2×2 differential systems are more interesting. If $\boldsymbol{x}' = A\,\boldsymbol{x}$ is such a system, it has fundamental solutions

$$\boldsymbol{x}^1(t) = \boldsymbol{v}\, e^{\lambda_0 t}, \qquad \boldsymbol{x}^2(t) = (\boldsymbol{v}\, t + \boldsymbol{w})\, e^{\lambda_0 t}, \tag{5.4.8}$$

where λ_0 is the repeated eigenvalue of A with eigenvector $\boldsymbol{v}$, and vector $\boldsymbol{w}$ is any solution of the linear algebraic system

$$(A - \lambda_0 I)\boldsymbol{w} = \boldsymbol{v}.$$

The phase portrait of these fundamental solutions is given in Fig 8. To construct this figure start drawing the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. The solution $\boldsymbol{x}^1$ is simpler to draw than $\boldsymbol{x}^2$, since the former is a straight semi-line starting at the origin and parallel to $\boldsymbol{v}$.

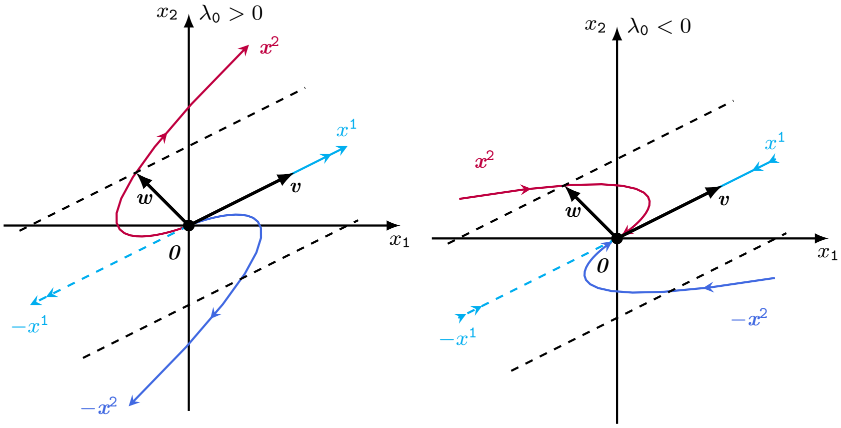


FIGURE 8. Functions $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ in Eq. (5.4.8) for the cases $\lambda_0 > 0$ and $\lambda_0 < 0$.

The solution $\boldsymbol{x}^2$ is more difficult to draw. One way is to first draw the trajectory of the time-dependent vector

$$\tilde{\boldsymbol{x}}^2 = \boldsymbol{v}\,t + \boldsymbol{w}.$$

This is a straight line parallel to $\boldsymbol{v}$ passing through $\boldsymbol{w}$, one of the black dashed lines in Fig. 8, the one passing through $\boldsymbol{w}$. The solution $\boldsymbol{x}^2$ differs from $\tilde{\boldsymbol{x}}^2$ by the multiplicative factor $e^{\lambda_0 t}$. Consider the case $\lambda_0 > 0$. For $t > 0$ we have $\boldsymbol{x}^2(t) > \tilde{\boldsymbol{x}}^2(t)$, and the opposite happens for $t < 0$. In the limit $t \to -\infty$ the solution values $\boldsymbol{x}^2(t)$ approach the origin, since the exponential factor $e^{\lambda_0 t}$ decreases faster than the linear factor t increases. The result is the purple line in the first picture of Fig. 8. The other picture, for $\lambda_0 < 0$ can be constructed following similar ideas.

5.4.4. Exercises.

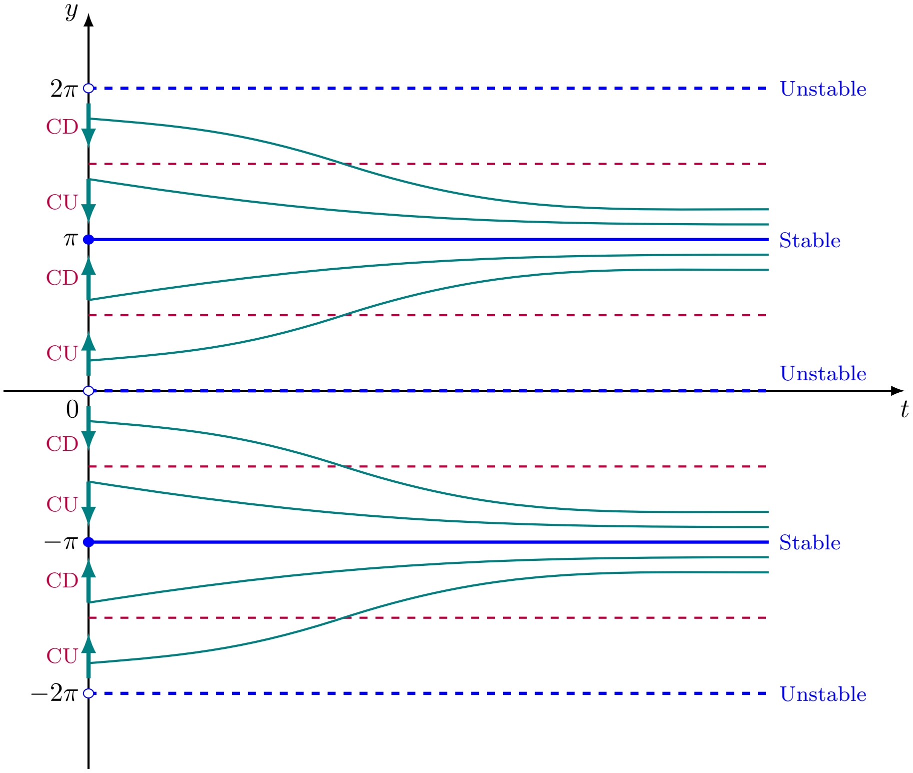
5.4.1.- .

5.4.2.- .

CHAPTER 6

Autonomous Systems and Stability

By the end of the seventeenth century Newton had invented differential equations, discovered his laws of motion and the law of universal gravitation. He combined all of them to explain Kepler laws of planetary motion. Newton solved what now is called the two-body problem. Kepler laws correspond to the case of one planet orbiting the Sun. People then started to study the three-body problem. For example the movement of Earth, Moon, and Sun. This problem turned out to be far more difficult than the two-body problem and no solution was ever found. Around the end of the nineteenth century Henri Poincaré proved a breakthrough result. The solutions of the three body problem could not be found explicitly in terms of elementary functions, such as combinations of polynomials, trigonometric functions, exponential, and logarithms. This led him to invent the so-called Qualitative Theory of Differential Equations. In this theory one studies the geometric properties of solutions–whether they show periodic behavior, tend to fixed points, tend to infinity, etc. This approach evolved into the modern field of Dynamics. In this chapter we introduce a few basic concepts and we use them to find qualitative information of a particular type of differential equations, called autonomous equations.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6.1. Flows on the Line

This chapter is dedicated to study the qualitative behavior of solutions to differential equations without actually computing the explicit expression of these solutions. In this section we focus on first order differential equations in one unknown function. We already have studied these equations in Chapter 1, § 1.1-1.4, and we have found formulas for their solutions. In this section we use these equations to present a new method to study qualitative properties of their solutions. Knowing the exact solution to the equation will help us understand how this new method works. In the next section we generalize the ideas in this section to 2×2 systems of nonlinear differential equations.

6.1.1. Autonomous Equations.

Let us study, one more time, first order nonlinear differential equations. In § 1.3 we learned how to solve separable equations–we integrated on both sides of the equation. We got an implicit expression for the solution in terms of the antiderivative of the equation coefficients. In this section we concentrate on a particular type of separable equations, called autonomous, where the independent variable does not appear explicitly in the equation. For these systems we find a few qualitative properties of their solutions without actually computing the solution. We find these properties of the solutions by studying the equation itself.

Definition 6.1.1. *A first order* ***autonomous*** *differential equation is*

$$y' = f(y), \tag{6.1.1}$$

where $y' = \dfrac{dy}{dt}$*, and the function* f *does not depend explictly on* t*.*

Remarks: The equation in (6.1.1) is a particular case of a separable equation where the independent variable t does not appear in the equation. This is the case, since Eq. (6.1.1) has the form

$$h(y)\, y' = g(t),$$

as in Def. 1.3.1, with $h(y) = 1/f(y)$ and $g(t) = 1$.

The autonomous equations we study in this section are a particular type of the separable equations we studied in § 1.3, as we can see in the following examples.

Example 6.1.1. The following first order separable equations are autonomous:

(a) $y' = 2\,y + 3$.

(b) $y' = \sin(y)$.

(c) $y' = r\,y\left(1 - \dfrac{y}{K}\right)$.

The independent variable t does not appear explicitly in these equations. The following equations are not autonomous.

(a) $y' = 2\,y + 3t$.

(b) $y' = t^2 \sin(y)$.

(c) $y' = t\,y\left(1 - \dfrac{y}{K}\right)$. ◁

Remark: Since the autonomous equation in (6.1.1) is a particular case of the separable equations from § 1.3, the Picard-Lindelöf Theorem applies to autonomous equations. Therefore, the initial value problem $y' = f(y)$, $y(0) = y_0$, with f continuous, always has a unique solution in the neighborhood of $t = 0$ for every value of the initial data y_0.

Sometimes an autonomous equation can be solved explicitly, with solutions simple to graph and simple to understand. Here is a well known example.

Example 6.1.2. Find all solutions of the first order autonomous system

$$y' = a\,y + b, \qquad a,\, b > 0.$$

Solution:

This is a linear, constant coefficients equation, so it could be solved using the integrating factor method. But this is also a separable equation, so we solve it as follows,

$$\int \frac{dy}{a\,y+b} = \int dt \;\Rightarrow\; \frac{1}{a}\,\ln(a\,y+b) = t + c_0$$

so we get,

$$a\,y + b = e^{at} e^{ac_0}$$

and denoting $c = e^{ac_0}/a$, we get the expression

$$y(t) = c\,e^{at} - \frac{b}{a}. \tag{6.1.2}$$

This is the expression for the solution we got in Theorem 1.1.2. ◁

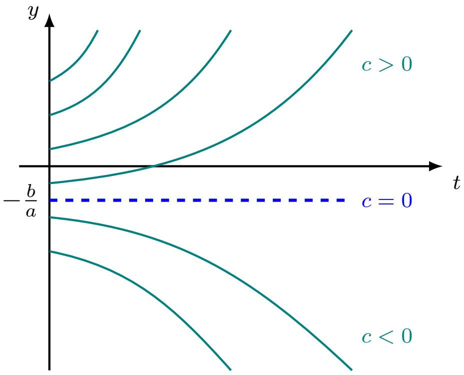


FIGURE 1. A few solutions to Eq. (6.1.2) for different c.

However, sometimes it is not so simple to grasp the qualitative behavior of solutions of an autonomous equation. Even in the case that we can solve the differential equation.

Example 6.1.3. Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution: We first find the exact solutions and then we see if we can graph them. The equation is separable, then

$$\frac{y'(t)}{\sin\big(y(t)\big)} = 1 \quad \Rightarrow \quad \int_0^t \frac{y'(t)}{\sin\big(y(t)\big)}\,dt = t.$$

Use the usual substitution $u = y(t)$, so $du = y'(t)\,dt$, so we get

$$\int_{y_0}^{y(t)} \frac{du}{\sin(u)} = t.$$

In an integration table we can find that

$$\ln\Big[\frac{\sin(u)}{1+\cos(u)}\Big]\Big|_{y_0}^{y(t)} = t \quad \Rightarrow \quad \ln\Big[\frac{\sin(y)}{1+\cos(y)}\Big] - \ln\Big[\frac{\sin(y_0)}{1+\cos(y_0)}\Big] = t.$$

We can rewrite the expression above in terms of one single logarithm,

$$\ln\Big[\frac{\sin(y)}{\big(1+\cos(y)\big)}\,\frac{\big(1+\cos(y_0)\big)}{\sin(y_0)}\Big] = t.$$

If we compute the exponential on both sides of the equation above we get an implicit expression of the solution,

$$\frac{\sin(y)}{\big(1+\cos(y)\big)} = \frac{\sin(y_0)}{\big(1+\cos(y_0)\big)}\,e^{t}. \tag{6.1.3}$$

Although we have the solution, in this case in implicit form, it is not simple to graph that solution without the help of a computer. So, we do not sketch the graph right now. ◁

It is not so easy to see certain properties of the solution from the exact expression in (6.1.3). For example, what is the behavior of the solution values $y(t)$ as $t \to \infty$ for an arbitrary initial condition y_0? To be able to answer questions like this one, is that we introduce a new approach, a geometric approach.

6.1.2. Geometrical Characterization of Stability. The idea is to obtain qualitative information about solutions to an autonomous equation using the equation itself, without solving it. We now use the equation in Example 6.1.3 to show how this can be done.

Example 6.1.4. Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0)$.

Solution: The differential equation has the form $y' = f(y)$, where $f(y) = \sin(y)$. The first step in the graphical approach is to graph the function f.

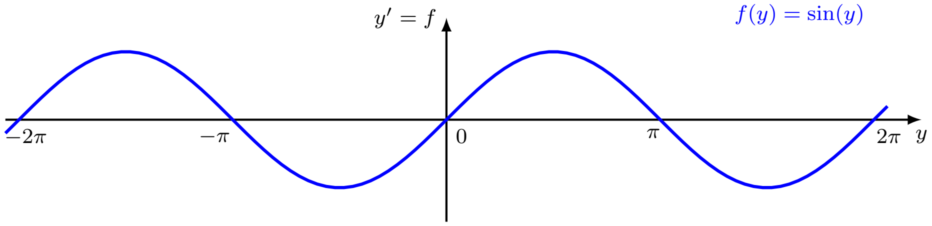


FIGURE 2. Graph of the function $f(y) = \sin(y)$.

The second step is to identify all the zeros of the function f. In this case,

$$f(y) = \sin(y) = 0 \quad \Rightarrow \quad y_n = n\pi, \quad \text{where} \quad n = \cdots, -2, -1, 0, 1, 2, \cdots.$$

It is important to realize that these constants y_n are solutions of the differential equation. On the one hand, they are constants, t-independent, so $y_n' = 0$. On the other hand, these constants y_n are zeros of f, hence $f(y_n) = 0$. So y_n are solutions of the differential equation

$$0 = y_n' = f(y_n) = 0.$$

The constants y_n are called *critical points*, or *fixed points*. When the emphasis is on the fact that these constants define constant functions solutions of the differential equation, then they are called *stationary solutions*, or *equilibrium solutions*.

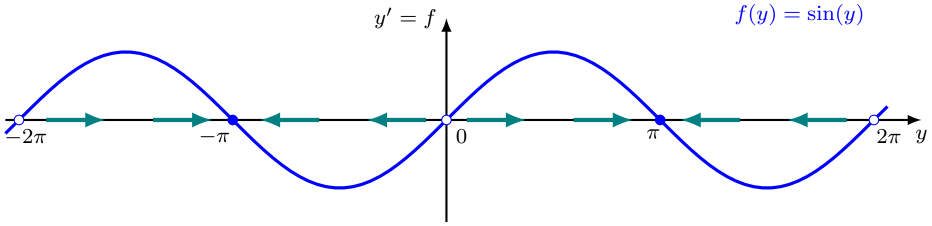


FIGURE 3. Critical points and increase/decrease information added to Fig. 2.

The third step is to identify the regions on the line where f is positive, and where f is negative. These regions are bounded by the critical points. A solution y of $y' = f(y)$ is increasing for $f(y) > 0$, and it is decreasing for $f(y) < 0$. We indicate this behavior of the solution by drawing arrows on the horizontal axis. In an interval where $f > 0$ we write a right arrow, and in the intervals where $f < 0$ we write a left arrow, as shown in Fig. 3.

There are two types of critical points in Fig. 3. The points $y_{\text{-}1} = -\pi$, $y_1 = \pi$, have arrows on both sides pointing to them. They are called *attractors*, or stable points, and are pictured with solid blue dots. The points $y_{\text{-}2} = -2\pi$, $y_0 = 0$, $y_2 = 2\pi$, have arrows on both sides pointing away from them. They are called *repellers*, or unstable points, and are pictured with white dots.

The fourth step is to find the regions where the curvature of a solution is concave up or concave down. That information is given by $y'' = (y')' = \big(f(y)\big)' = f'(y)\, y' = f'(y)\, f(y)$. So, in the regions where $f(y)\, f'(y) > 0$ a solution is concave up (CU), and in the regions where $f(y)\, f'(y) < 0$ a solution is concave down (CD). See Fig. 4.

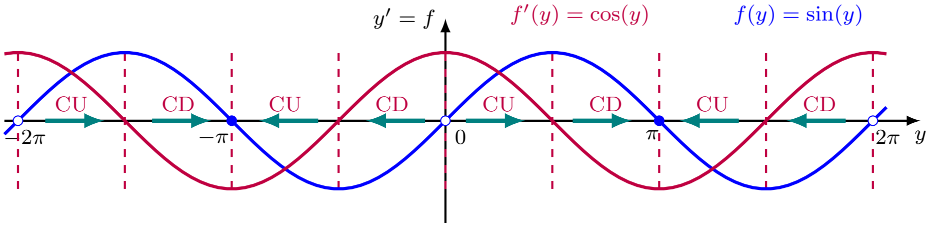


FIGURE 4. Concavity information on the solution y added to Fig. 3.

This is all we need to sketch a qualitative graph of solutions to the differential equation. The last step is to collect all this information on a ty-plane. The horizontal axis above is now the vertical axis, and we now plot soltuions y of the differential equation. See Fig. 5.

Fig. 5 contains the graph of several solutions y for different choices of initial data $y(0)$. Stationary solutions are in blue, t-dependent solutions in green. The stationary solutions are separated in two types. The stable solutions $y_{\text{-}1} = -\pi$, $y_1 = \pi$, are pictured with solid blue lines. The unstable solutions $y_{\text{-}2} = -2\pi$, $y_0 = 0$, $y_2 = 2\pi$, are pictured with dashed blue lines. $\lhd$

Remark: A qualitative graph of the solutions does not provide all the possible information about the solution. For example, we know from the graph above that for some initial conditions the corresponding solutions have inflection points at some $t > 0$. But we cannot know the exact value of t where the inflection point occurs. Such information could be useful to have, since $|y'|$ has its maximum value at those points.

In the Example 6.1.4 above we have used that the second derivative of the solution function is related to f and f'. This is a result that we remark here in its own statement.

Theorem 6.1.2. *If y is a solution of the autonomous system $y' = f(y)$, then*

$$y'' = f'(y)\, f(y).$$

Remark: This result has been used to find out the curvature of the solution y of an autonomous system $y' = f(y)$. The graph of y has positive curvature iff $f'(y)\, f(y) > 0$ and negative curvature iff $f'(y)\, f(y) < 0$.

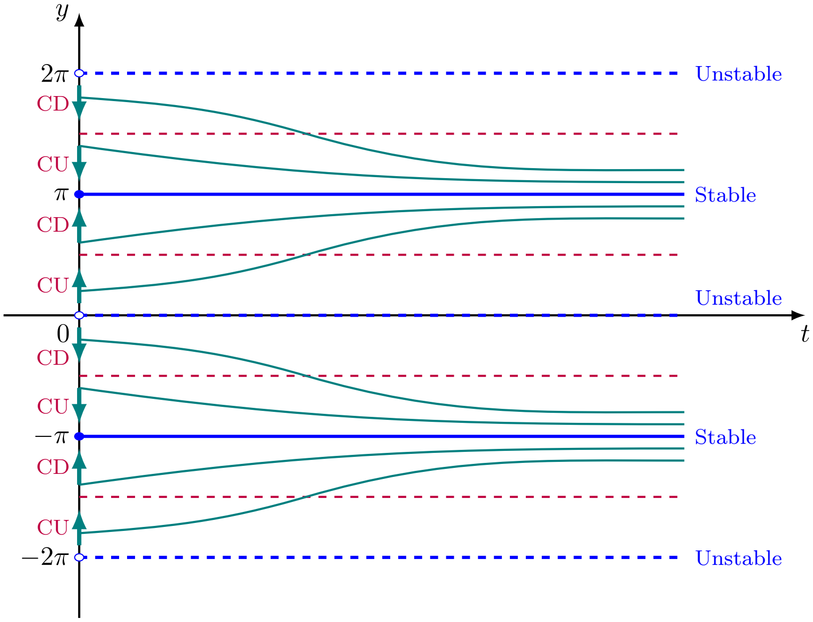


FIGURE 5. Qualitative graphs of solutions y for different initial conditions.

Proof: The differential equation relates y'' to $f(y)$ and $f'(y)$, because of the chain rule,

$$y'' = \frac{d}{dt}\Big(\frac{dy}{dt}\Big) = \frac{d}{dt} f(y(t)) = \frac{df}{dy}\,\frac{dy}{dt} \quad \Rightarrow \quad y'' = f'(y)\, f(y).$$

□

6.1.3. Critical Points and Linearization.

Let us summarize a few definitions we introduced in the Example 6.1.3 above.

Definition 6.1.3.

*A point y_c is a **critical point** of $y' = f(y)$ iff $f(y_c) = 0$. A critical points is:*

*(i) an **attractor** (or sink), iff solutions flow toward the critical point;*
*(ii) a **repeller** (or source), iff solutions flow away from the critical point;*
*(iii) **neutral**, iff solution flow towards the critical point from one side and flow away from the other side.*

In this section we keep the convention used in the Example 6.1.3, filled dots denote attractors, and white dots denote repellers. We will use a a half-filled point for neutral points. We recall that attractors have arrows directed to them on both sides, while repellers have arrows directed away from them on both sides. A neutral point would have an arrow pointing towards the critical point on one side and the an arrow pointing away from the critical point on the other side. We will usually mention critical points as stationary solutions when we describe them in a yt-plane, and we reserve the name critical point when we describe them in the phase line, the y-line.

We also talked about stable and unstable solutions. Here is a precise definition.

Definition 6.1.4. *Let y_0 be a a constant solution of $y' = f(y)$, and let y be a solution with initial data $y(0) = y_1$. The solution given by y_0 is* ***stable*** *iff given any $\epsilon > 0$ there is a $\delta > 0$ such that if the initial data y_1 satisfies $|y_1 - y_0| < \delta$, then the solution values $y(t)$ satisfy $|y(t) - y_0| < \epsilon$ for all $t > 0$. Furthermore, if $\lim_{t\to\infty} y(t) = y_0$, then y_0 is* ***asymptotically stable****. If y_0 is not stable, we call it* ***unstable****.*

The geometrical method described in Example 6.1.3 above is useful to get a quick qualitative picture of solutions to an autonomous differential system. But it is always nice to complement geometric methods with analytic methods. For example, one would like an analytic way to determine the stability of a critical point. One would also like a quantitative measure of a solution decay rate to a stationary solution. A linear stability analysis can provide this type of information.

We start assuming that the function f has a Taylor expansion at any y_0. That is,

$$f(y) = f(y_0) + f'(y_0)\,(y - y_0) + o((y - y_0)^2).$$

Denote $f_0 = f(y_0)$, then $f_0' = f'(y_0)$, and introduce the variable $u = y - y_0$. Then we get

$$f(y) = f_0 + f_0'\,u + o(u^2).$$

Let us use this Taylor expansion on the right hand side of the equation $y' = f(y)$, and recalling that $y' = (y_0 + u)' = u'$, we get

$$y' = f(y) \quad \Leftrightarrow \quad u' = f_0 + f_0'\,u + o(u^2).$$

If y_0 is a critical point of f, then $f_0 = 0$, then

$$y' = f(y) \quad \Leftrightarrow \quad u' = f_0'\,u + o(u^2).$$

From the equations above we see that for $y(t)$ close to a critical point y_0 the right hand side of the equation $y' = f(y)$ is close to $f_0'\,u$. Therefore, one can get information about a solution of a nonlinear equation near a critical point by studying an appropriate linear equation. We give this linear equation a name.

Definition 6.1.5. *The* ***linearization*** *of an autonomous system $y' = f(y)$ at a critical point y_c is the linear differential equation for the function u given by*

$$u' = f'(y_c)\,u.$$

Remark: The prime notation above means, $u' = du/dt$, and $f' = df/dy$.

Example 6.1.5. Find the linearization of the equation $y' = \sin(y)$ at the critical point $y_n = n\pi$. Write the particular cases for $n = 0, 1$ and solve the linear equations for arbitrary initial data.

Solution: If we write the nonlinear system as $y' = f(y)$, then $f(y) = \sin(y)$. We then compute its y derivative, $f'(y) = \cos(y)$. We evaluate this expression at the critical points, $f'(y_n) = \cos(n\pi) = (-1)^n$. The linearization at y_n of the nonlinear equation above is the linear equation for the unknown function u_n given by

$$u_n' = (-1)^n\,u_n.$$

The particular cases $n = 0$ and $n = 1$ are given by

$$u_0' = u_0, \qquad u_1' = -u_1.$$

It is simple to find solutions to first order linear homogeneous equations with constant coefficients. The result, for each equation above, is

$$u_0(t) = u_0(0)\,e^{t}, \qquad u_1(t) = u_1(0)\,e^{-t}.$$

◁

As we see in the the Def. 6.1.5 and in Example 6.1.5, the linearization of $y' = f(y)$ at a critical point y_0 is quite simple, it is the linear equation $u' = a\,u$, where $a = f'(y_0)$. We know all the solutions to this linear equation, we computed them in § 1.1.

Theorem 6.1.6 (Stability of Linear Equations). *The constant coefficent linear equation $u' = a\,u$, with $a \neq 0$, has only one critical point $u_0 = 0$. And the constant solution defined by this critical point is unstable for $a > 0$, and it is asymptotically stable for $a < 0$.*

Proof of Theorem 6.1.6: The critical points of the linear equation $u' = a\,u$ are the solutions of $au = 0$. Since $a \neq 0$, that means we have only one critical point, $u_0 = 0$. Since the linear equation is so simple to solve, we can study the stability of the constant solution $u_0 = 0$ from the formula for all the solutions of the equation,

$$u(t) = u(0)\,e^{at}.$$

The graph of all these solutions is sketch in Fig. 6. in the case that $u(0) \neq 0$, we see that for $a > 0$ the solutions diverge to ±∞ as $t \to \infty$, and for $a < 0$ the solutions approach to zero as $t \to \infty$. ☐

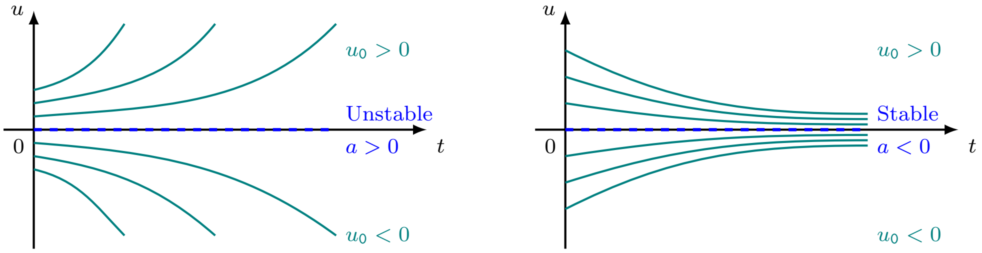


FIGURE 6. The graph of the functions $u(t) = u(0)\,e^{at}$ for $a > 0$ and $a < 0$.

Remark: In the Example 6.1.5 above (and later on in Example 6.1.8) we see that the stability of a critical point y_c of a nonlinear differential equation $y' = f(y)$ is the same as the stability of the trivial solution $u = 0$ of the linearization $u' = f'(y_c)\,u$. This is a general result, which we state below.

Theorem 6.1.7 (Stability of Nonlinear Equations). *Let y_c be a critical point of the autonomous system $y' = f(y)$.*

(a) The critical point y_c is stable iff $f'(y_c) < 0$.
(b) The critical point y_c is unstable iff $f'(y_c) > 0$.

Furthermore, If the initial data $y(0) \simeq y_c$, is close enough to the critial point y_c, then the solution with that initial data of the equation $y' = f(y)$ are close enough to y_c in the sense

$$y(t) \simeq y_c + u(t),$$

where u is the solution to the linearized equation at the critical point y_c,

$$u' = f'(y_c)\,u, \qquad u(0) = y(0) - y_c.$$

Remark: The proof of this result can be found in § 2.4 in Strogatz textbook [12].

Remark: The first part of Theorem 6.1.7 highlights the importance of the sign fo the coefficient $f'(y_c)$, which determines the stability of the critical point y_c. The furthermore part of the Theorem highlights how stable is a critical point. The value $|f'(y_c)|$ plays a role of an exponential growth or a exponential decay rate. Its reciprocal, $1/|f'(y_c)|$ is a *characteristic scale*. It determines the value of t required for the solution y to vary significantly in a neighborhood of the critical point y_c.

6.1.4. Population Growth Models.

The simplest model for the population growth of an organism is $N' = rN$ where $N(t)$ is the population at time t and $r > 0$ is the growth rate. This model predicts exponential population growth $N(t) = N_0\, e^{rt}$, where $N_0 = N(0)$. This model assumes that the organisms have unlimited food supply, hence the per capita growth $N'/N = r$ is constant.

A more realistic model assumes that the per capita growth decreases linearly with N, starting with a positive value, r, and going down to zero for a critical population $N = K > 0$. So when we consider the per capita growth N'/N as a function of N, it must be given by the formula $N'/N = -(r/K)N + r$. This is the logistic model for population growth.

Definition 6.1.8. *The* ***logistic equation*** *describes the organisms population function N in time as the solution of the autonomous differential equation*

$$N' = rN\left(1 - \frac{N}{K}\right),$$

where the initial growth rate constant r and the carrying capacity constant K are positive.

Remark: The logistic equation is, of course, a separable equation, so it can be solved using the method from § 1.3. We solve it below, so you can compare the qualitative graphs from Example 6.1.7 with the exact solution below.

Example 6.1.6. Find the exact expression for the solution to the logistic equation for population growth

$$y' = ry\left(1 - \frac{y}{K}\right), \qquad y(0) = y_0, \qquad 0 < y_0 < K.$$

Solution: This is a separable equation,

$$\frac{K}{r}\int \frac{y'\,dt}{(K-y)y} = t + c_0.$$

The usual substitution $u = y(t)$, so $du = y'\,dt$, implies

$$\frac{K}{r}\int \frac{du}{(K-u)u} = t + c_0. \quad\Rightarrow\quad \frac{K}{r}\int \frac{1}{K}\Big[\frac{1}{(K-u)} + \frac{1}{u}\Big]\,du = t + c_0.$$

where we used partial fractions decomposistion to get the second equation. Now, each term can be integrated,

$$\big[-\ln(|K-y|) + \ln(|y|)\big] = rt + rc_0.$$

We reorder the terms on the right-hand side,

$$\ln\Big(\frac{|y|}{|K-y|}\Big) = rt + rc_0 \quad\Rightarrow\quad \Big|\frac{y}{K-y}\Big| = c\,e^{rt}, \quad c = e^{rc_0}.$$

The analysis done in Example 6.1.4 says that for initial data $0 < y_0 < K$ we can discard the absolute values in the expression above for the solution. Now the initial condition fixes the value of the constant c,

$$\frac{y_0}{K-y_0} = c \quad\Rightarrow\quad y(t) = \frac{Ky_0}{y_0 + (K-y_0)\,e^{-rt}}.$$

◁

Remark: The expression above provides all solutions to the logistic equation with initial data on the interval $(0, K)$. With some more work one could graph these solution and get a picture of the solution behaviour. We now use the graphical method discussed above to get a qualitative picture of the solution graphs without solving the differential equation.

Example 6.1.7. Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the logistic equation below, where r and K are given positive constants,

$$y' = ry\left(1 - \frac{y}{K}\right).$$

Solution:
The logistic differential equation for population growth can be written $y' = f(y)$, where function f is the polynomial

$$f(y) = ry\left(1 - \frac{y}{K}\right).$$

The first step in the graphical approach is to graph the function f. The result is in Fig. 7.

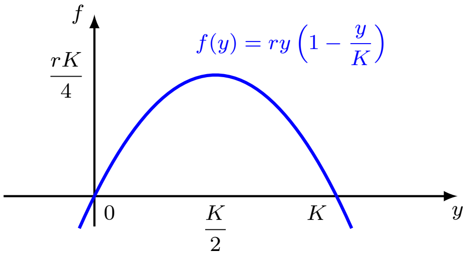


FIGURE 7. The graph of f.

The second step is to identify all critical points of the equation. The critical points are the zeros of the function f. In this case, $f(y) = 0$ implies

$$y_0 = 0, \quad y_1 = K.$$

The third step is to find out whether the critical points are stable or unstable. Where function f is positive, a solution will be increasing, and where function f is negative a solution will be decreasing. These regions are bounded by the critical points. Now, in an interval where $f > 0$ write a right arrow, and in the intervals where $f < 0$ write a left arrow, as shown in Fig. 8.

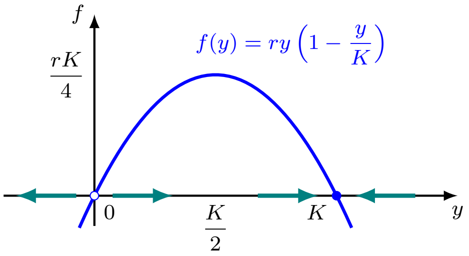


FIGURE 8. Critical points added.

This is all the information we need to sketch a qualitative graph of solutions to the differential equation. So, the last step is to put all this information on a yt-plane. The horizontal axis above is now the vertical axis, and we now plot solutions y of the differential equation. The result is given in Fig. 10.

The picture above contains the graph of several solutions y for different choices of initial data $y(0)$. Stationary solutions are in blue, t-dependent solutions in green. The stationary solution $y_0 = 0$ is unstable and pictured with a dashed blue line. The stationary solution $y_1 = K$ is stable and pictured with a solid blue line. ◁

Example 6.1.8. Find the linearization of the logistic equation $y' = ry\left(1 - \dfrac{y}{K}\right)$ at the critical points $y_0 = 0$ and $y_1 = K$. Solve the linear equations for arbitrary initial data.

The fourth step is to find the regions where the curvature of a solution is concave up or concave down. That information is given by y''. But the differential equation relates y'' to $f(y)$ and $f'(y)$. We have shown in Example 6.1.4 that the chain rule and the differential equation imply,

$$y'' = f'(y)\, f(y)$$

So the regions where $f(y)\, f'(y) > 0$ a solution is concave up (CU), and the regions where $f(y)\, f'(y) < 0$ a solution is concave down (CD). The result is in Fig. 9.

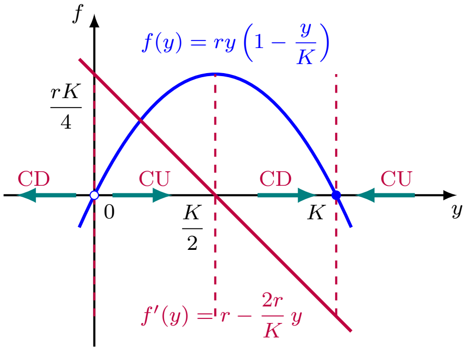


Figure 9. Concavity information added.

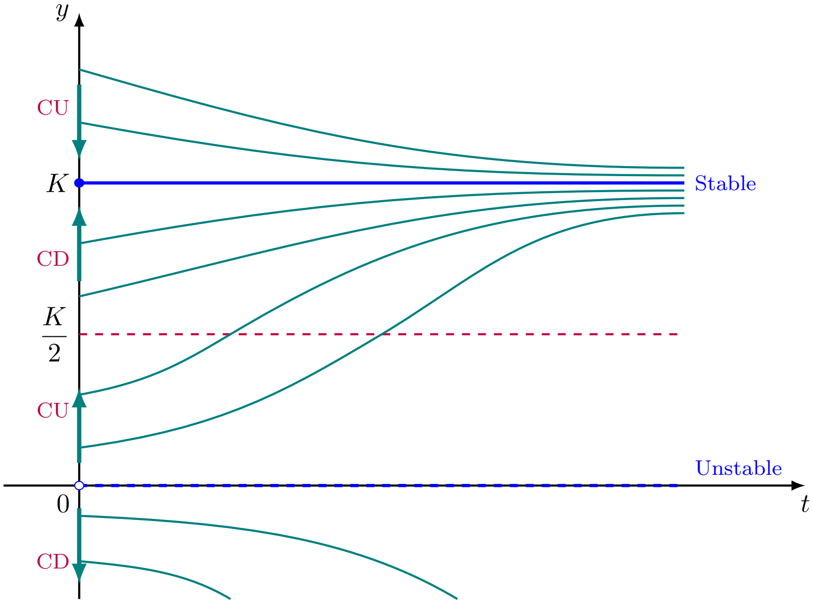


Figure 10. Qualitative graphs of solutions y for different initial conditions.

Solution: If we write the nonlinear system as $y' = f(y)$, then $f(y) = ry\Big(1 - \frac{y}{K}\Big)$. The critical points are $y_0 = 0$ and $y_1 = K$. We also need to compute $f'(y) = r - \frac{2r}{K}\, y$. For the critical point $y_0 = 0$ we get the linearized system

$$u_0'(t) = r\, u_0 \quad \Rightarrow \quad u_0(t) = u_0(0)\, e^{rt}.$$

For the critical point $y_1 = K$ we get the linearized system

$$u_1'(t) = -r\, u_1 \quad \Rightarrow \quad u_1(t) = u_1(0)\, e^{-rt}.$$

From this last expression we can see that for $y_0 = 0$ the critical solution $u_0 = 0$ is unstable, while for $y_1 = K$ the critical solution $u_1 = 0$ is stable. The stability of the trivial solution

$u_0 = u_1 = 0$ of the linearized system coincides with the stability of the critical points $y_0 = 0$, $y_1 = K$ for the nonlinear equation. $\lhd$

Notes

This section follows a few parts of Chapter 2 in Steven Strogatz's book on Nonlinear Dynamics and Chaos, [12], and also § 2.5 in Boyce DiPrima classic textbook [3].

6.1.5. Exercises.

6.1.1.- .

6.1.2.- .

6.2. Flows on the Plane

We now turn to study two-dimensional *nonlinear* autonomous systems. We start reviewing the critical points of two-by-two linear systems and classifying them as attractors, repellers, centers, and saddle points. We then introduce a few examples of two-by-two nonlinear systems. We define the critical points of nonlinear systems. We then compute the linearization of these systems and we study the linear stability of these two-dimensional nonlinear systems. In the next section we solve a few examples from biology (predator-prey systems and competing species systems), and from physics (the nonlinear pendulum and potential systems).

6.2.1. Two-Dimensional Nonlinear Systems.

We start with the definition of autonomous systems on the plane.

Definition 6.2.1. *A first order **two-dimensional autonomous** differential equation is*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}),$$

where $\boldsymbol{x}' = \dfrac{d\boldsymbol{x}}{dt}$*, and the vector field* $\boldsymbol{f}$ *does not depend explicitly on* t*.*

Remark: If we introduce the vector components $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$, then the autonomous equation above can be written in components,

$$x_1' = f_1(x_1, x_2),$$
$$x_2' = f_2(x_1, x_2),$$

where $x_i' = \dfrac{dx_i}{dt}$, for $i = 1, 2$.

Example 6.2.1 (The Nonlinear Pendulum)**.** A pendulum of mass m, length ℓ, oscillating under the gravity acceleration g, moves according to Newton's second law of motion

$$m(\ell\theta)'' = -mg\sin(\theta),$$

where the angle θ depends on time t. If we rearrange terms we get a second order scalar equation

$$\theta'' + \frac{g}{\ell}\sin(\theta) = 0.$$

This scalar equation can be written as a nonlinear system. If we introduce $x_1 = \theta$ and $x_2 = \theta'$, then

$$x_1' = x_2$$
$$x_2' = -\frac{g}{\ell}\sin(x_1).$$

◁

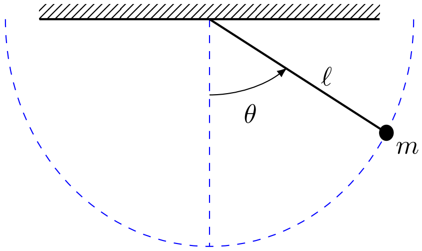


FIGURE 11. Pendulum.

Example 6.2.2 (Predator-Prey)**.** The physical system consists of two biological species where one preys on the other. For example cats prey on mice, foxes prey on rabbits. If we call x_1 the predator population, and x_2 the prey population, then predator-prey equations, also known as Lotka-Volterra equations for predator prey, are

$$\begin{aligned} x_1' &= -a\,x_1 + b\,x_1x_2, \\ x_2' &= -c\,x_1x_2 + d\,x_2. \end{aligned}$$

The constants a, b, c, d are all nonnegative. Notice that in the case of absence of predators, $x_1 = 0$, the prey population grows without bounds, since $x_2' = d\,x_2$. In the case of absence of prey, $x_2 = 0$, the predator population becames extinct, since $x_1' = -a\,x_1$. The term $-c\,x_1x_2$ represents the prey death rate due to predation, which is porportional to the number of encounters, x_1x_2, between predators and prey. These encounters have a positive contribution $b\,x_1x_2$ to the predator population. $\lhd$

Example 6.2.3 (Competing Species)**.** The physical system consists of two species that compete on the same food resources. For example, rabbits and sheep, which compete on the grass on a particular piece of land. If x_1 and x_2 are the competing species popultions, the the differential equations, also called Lotka-Volterra equations for competition, are

$$\begin{aligned} x_1' &= r_1\,x_1\Big(1 - \frac{x_1}{K_1} - \alpha\,x_2\Big), \\ x_2' &= r_2\,x_2\Big(1 - \frac{x_2}{K_2} - \beta\,x_1\Big). \end{aligned}$$

The constants r_1, r_2, α, β are all nonnegative, and K_1, K_2 are positive. Note that in the case of absence of one species, say $x_2 = 0$, the population of the other species, x_1 is described by a logistic equation. The terms $-\alpha\,x_1x_2$ and $-\beta\,x_1x_2$ say that the competition between the two species is proportional to the number of competitive pairs x_1x_2. $\lhd$

6.2.2. Review: The Stability of Linear Systems. In § **??** we used phase portraits to display vector functions

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

solutions of 2×2 linear differential systems. In a phase portrait we plot the vector $\boldsymbol{x}(t)$ on the plane x_1x_2 for different values of the independent variable t.We then plot a curve representing all the end points of the vectors $\boldsymbol{x}(t)$, for t on some interval. The arrows in the curve show the direction of increasing t.

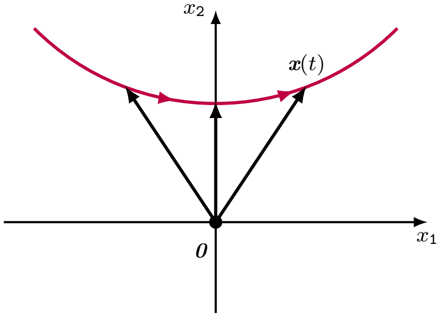


FIGURE 12. A curve in a phase portrait represents all the end points of the vectors $\boldsymbol{x}(t)$, for t on some interval. The arrows in the curve show the direction of increasing t.

We saw in § **??** that the behavior of solutions to 2-dimensional linear systems depend on the eigenvalues of the coefficient matrix. If we denote a general 2×2 matrix by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then the eigenvalues are the roots of the characteristic polynomial,

$$\det(A - \lambda I) = \lambda^2 - p\,\lambda + q = 0,$$

where we denoted $p = a_{11} + a_{22}$ and $q = a_{11}a_{22} - a_{12}a_{21}$. Then the eigenvalues are

$$\lambda_{\pm} = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{p}{2} \pm \frac{\sqrt{\Delta}}{2},$$

where $\Delta = p^2 - 4q$. We can classify the eigenvalues according to the sign of Δ. In Fig 13 we plot on the pq-plane the curve $\Delta = 0$, that is, the parabola $q = p^2/4$. The region above this parabola is $\Delta < 0$, therefore the matrix eigenvalues are complex, which corresponds to spirals in the phase portrait. The spirals are stable for $p < 0$ and unstable for $p > 0$. The region below the parabola corresponds to real disctinct eigenvalues. The parabola itself corresponds to the repeated eigenvalue case.

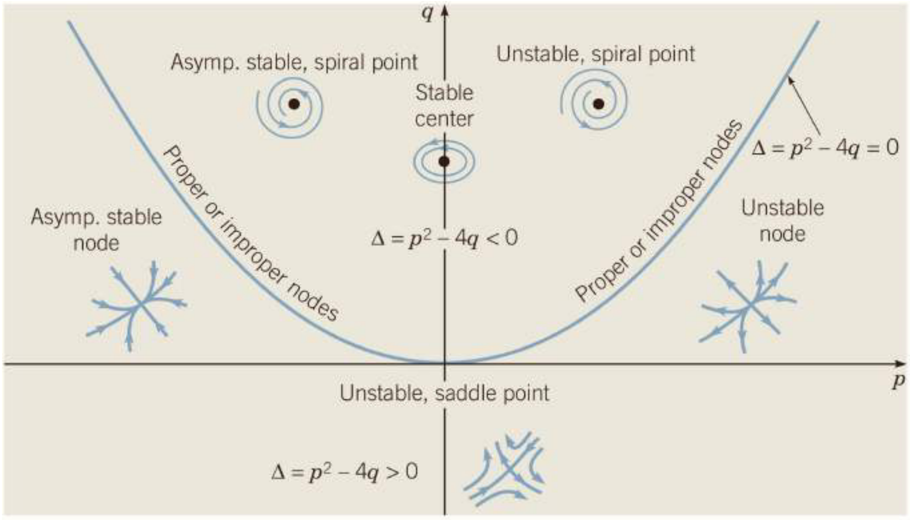


Figure 13. The stability of the solution $\boldsymbol{x}_0 = \boldsymbol{0}$. (Boyce DiPrima, § 9.1, [3].)

The trivial solution $\boldsymbol{x}_0 = \boldsymbol{0}$ is called a *critical point* of the linear system $\boldsymbol{x}' = A\boldsymbol{x}$. Here is a more detailed classification of this critical point.

Definition 6.2.2. *The critical point $\boldsymbol{x}_0 = \mathbf{0}$ of a 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$ is:*

*(a) an **attractor** (or sink), iff both eigenvalues of A have negative real part;*
*(b) a **repeller** (or source), iff both eigenvalues of A have positive real part;*
*(c) a **saddle**, iff one eigenvalue of A is positive and the other is negative;*
*(d) a **center**, iff both eigenvalues of A are pure imaginary;*
*(e) **higher order** critical point iff at least one eigenvalue of A is zero.*

*The critical point $\boldsymbol{x}_0 = \mathbf{0}$ is called **hyperbolic** iff it belongs to cases (a-c), that is, the real part of all eigenvalues of A are nonzero.*

We saw in § **??** that the behavior of solutions to a linear system $\boldsymbol{x}' = A\boldsymbol{x}$, with initial data $\boldsymbol{x}(0)$, depends on what type of critical point is $\boldsymbol{x}_0 = \boldsymbol{0}$. The results presented in that section can be summarized in the following statement.

Theorem 6.2.3 (Stability of Linear Systems). *Let $\boldsymbol{x}(t)$ be the solution of a 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, with $\det(A) \neq 0$ and initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_1$.*

- *(a) The critical point $\boldsymbol{x}_0 = \mathbf{0}$ is an attractor iff for any initial condition $\boldsymbol{x}(0)$ the corresponding solution $\boldsymbol{x}(t)$ satisfies that $\lim_{t\to\infty} \boldsymbol{x}(t) = \mathbf{0}$.*
- *(b) The critical point $\boldsymbol{x}_0 = \mathbf{0}$ is a repeller iff for any initial condition $\boldsymbol{x}(0)$ the corresponding solution $\boldsymbol{x}(t)$ satisfies that $\lim_{t\to\infty} |\boldsymbol{x}(t)| = \infty$.*
- *(c) The critical point $\boldsymbol{x}_0 = \mathbf{0}$ is a center iff for any initial data $\boldsymbol{x}(0)$ the corresponding solution $\boldsymbol{x}(t)$ describes a closed periodic trajectory around $\mathbf{0}$.*

Phase portraits will be very useful to understand solutions to 2-dimensional *nonlinear* differential equations. We now state the main result about solutions to autonomous systems $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is the following.

Theorem 6.2.4 (IVP). *If the field $\boldsymbol{f}$ differentiable on some open connected set $D \in \mathbb{R}^2$, then the initial value problem*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in D,$$

has a unique solution $\boldsymbol{x}(t)$ on some nonempty interval $(-t_1, t_1)$ about $t = 0$.

Remark: The fixed point argument used in the proof of Picard-Lindelöf's Theorem 1.6.2 can be extended to prove Theorem 6.2.4. This proof will be presented later on.

Remark: That the field $\boldsymbol{f}$ is differentiable on $D \in \mathbb{R}^2$ means that $\boldsymbol{f}$ is continuous, and all the partial derivatives $\partial f_i/\partial x_j$, for $i, j = 1, 2$, are continuous for all $\boldsymbol{x}$ in D.

Theorem 6.2.4 has an important corollary: *different trajectories never intersect.* If two trajectories did intersect, then there would be two solutions starting from the same point, the crossing point. This would violate the uniqueness part of the theorem. Because trajectories cannot intersect, phase portraits of autonomous systems have a well-groomed appearence.

6.2.3. Critical Points and Linearization.

We now extended to two-dimensional systems the concept of linearization we introduced for one-dimensional systems. The hope is that solutions to nonlinear systems close to critical points behave in a similar way to solutions to the linearized system. We will see that this is the case if the linearized system has distinct eigenvalues. Se start with the definition of critical points.

Definition 6.2.5. *A **critical point** of a two-dimensional system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is a vector $\boldsymbol{x}_0$ where the field $\boldsymbol{f}$ vanishes,*

$$\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}.$$

Remark: A critical point defines a constant vector function $\boldsymbol{x}(t) = \boldsymbol{x}_0$ for all t, solution of the differential equation,

$$\boldsymbol{x}_0' = \boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_0).$$

In components, the field is $\boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, and the critical point $\boldsymbol{x}_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$ is solution of

$$\begin{aligned} f_1(x_1^0, x_2^0) &= 0, \\ f_2(x_1^0, x_2^0) &= 0. \end{aligned}$$

When there is more than one critical point we will use the notation $\boldsymbol{x}_i$, with $i = 0, 1, 2, \cdots$, to denote the critical points.

Example 6.2.4. Find all the critical points of the two-dimensional (decoupled) system

$$\begin{aligned} x_1' &= -x_1 + (x_1)^3 \\ x_2' &= -2\,x_2. \end{aligned}$$

Solution: We need to find all constant vectors $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ solutions of

$$\begin{aligned} -x_1 + (x_1)^3 &= 0, \\ -2\,x_2 &= 0. \end{aligned}$$

From the second equation we get $x_2 = 0$. From the first equation we get

$$x_1\big((x_1)^2 - 1\big) = 0 \quad \Rightarrow \quad x_1 = 0, \quad \text{or} \quad x_1 = \pm 1.$$

Therefore, we got three critical points, $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. ◁

We now generalize to two-dimensional systems the idea of linearization introduced in § 6.1 for scalar equations. Consider the two-dimensional system

$$\begin{aligned} x_1' &= f_1(x_1, x_2), \\ x_2' &= f_2(x_1, x_2), \end{aligned}$$

Assume that f_1, f_2 have Taylor expansions at $\boldsymbol{x}_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$. We denote $u_1 = (x_1 - x_1^0)$ and $u_2 = (x_2 - x_2^0)$, and $f_1^0 = f_1(x_1^0, x_2^0)$, $f_2^0 = f_2(x_1^0, x_2^0)$. Then, by the Taylor expansion theorem,

$$\begin{aligned} f_1(x_1, x_2) &= f_1^0 + \frac{\partial f_1}{\partial x_1}\Big|_{\boldsymbol{x}_0} u_1 + \frac{\partial f_1}{\partial x_2}\Big|_{\boldsymbol{x}_0} u_2 + o\big(u_1^2, u_2^2\big), \\ f_2(x_1, x_2) &= f_2^0 + \frac{\partial f_2}{\partial x_1}\Big|_{\boldsymbol{x}_0} u_1 + \frac{\partial f_2}{\partial x_2}\Big|_{\boldsymbol{x}_0} u_2 + o\big(u_1^2, u_2^2\big). \end{aligned}$$

Let us simplify the notation a bit further. Let us denote

$$\begin{aligned} \partial_1 f_1 &= \frac{\partial f_1}{\partial x_1}\Big|_{\boldsymbol{x}_0}, & \partial_2 f_1 &= \frac{\partial f_1}{\partial x_2}\Big|_{\boldsymbol{x}_0}, \\ \partial_1 f_2 &= \frac{\partial f_2}{\partial x_1}\Big|_{\boldsymbol{x}_0}, & \partial_2 f_2 &= \frac{\partial f_2}{\partial x_2}\Big|_{\boldsymbol{x}_0}. \end{aligned}$$

then the Taylor expansion of $\boldsymbol{f}$ has the form

$$\begin{aligned} f_1(x_1, x_2) &= f_1^0 + (\partial_1 f_1)\,u_1 + (\partial_2 f_1)\,u_2 + o\big(u_1^2, u_2^2\big), \\ f_2(x_1, x_2) &= f_2^0 + (\partial_1 f_2)\,u_1 + (\partial_2 f_2)\,u_2 + o\big(u_1^2, u_2^2\big). \end{aligned}$$

We now use this Taylor expansion of the field $\boldsymbol{f}$ into the differential equation $\boldsymbol{x}' = \boldsymbol{f}$. Recall that $x_1 = x_1^0 + u_1$, and $x_2 = x_2^0 + u_2$, and that x_1^0 and x_2^0 are constants, then

$$\begin{aligned} u_1' &= f_1^0 + (\partial_1 f_1)\, u_1 + (\partial_2 f_1)\, u_2 + o\big(u_1^2, u_2^2\big),\\ u_2' &= f_2^0 + (\partial_1 f_2)\, u_1 + (\partial_2 f_2)\, u_2 + o\big(u_1^2, u_2^2\big). \end{aligned}$$

Let us write this differential equation using vector notation. If we introduce the vectors and the matrix

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \boldsymbol{f}_0 = \begin{bmatrix} f_1^0 \\ f_2^0 \end{bmatrix}, \qquad Df_0 = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix},$$

then, we have that

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) \quad \Leftrightarrow \quad \boldsymbol{u}' = \boldsymbol{f}_0 + (Df_0)\, \boldsymbol{u} + o\big(|\boldsymbol{u}|^2\big).$$

In the case that x_0 is a critical point, then $\boldsymbol{f}_0 = \boldsymbol{0}$. In this case we have that

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) \quad \Leftrightarrow \quad \boldsymbol{u}' = (Df_0)\, \boldsymbol{u} + o\big(|\boldsymbol{u}|^2\big).$$

The relation above says that the equation coefficients of $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ are close, order $o\big(|\boldsymbol{u}|^2\big)$, to the coefficients of the linear differential equation $\boldsymbol{u}' = (Df_0)\, \boldsymbol{u}$. For this reason, we give this linear differential equation a name.

Definition 6.2.6. *The* ***linearization*** *of a two-dimensional system* $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ *at a critical point* $\boldsymbol{x}_0$ *is the* 2×2 *linear system*

$$\boldsymbol{u}' = (Df_0)\, \boldsymbol{u},$$

where $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_0$*, and we have introduced the* ***Jacobian matrix*** *at* $\boldsymbol{x}_0$*,*

$$Df_0 = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\boldsymbol{x}_0} & \left.\frac{\partial f_1}{\partial x_2}\right|_{\boldsymbol{x}_0} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{\boldsymbol{x}_0} & \left.\frac{\partial f_2}{\partial x_2}\right|_{\boldsymbol{x}_0} \end{bmatrix} = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix}.$$

Remark: In components, the nonlinear system is

$$\begin{aligned} x_1' &= f_1(x_1, x_2),\\ x_2' &= f_2(x_1, x_2), \end{aligned}$$

and the linearization at $\boldsymbol{x}_0$ is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Example 6.2.5. Find the linearization at every critical point of the nonlinear system

$$\begin{aligned} x_1' &= -x_1 + (x_1)^3\\ x_2' &= -2\, x_2. \end{aligned}$$

Solution: We found earlier that this system has three critial points,

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

This means we need to compute three linearizations, one for each critical point. We start computing the derivative matrix at an arbitrary point $\boldsymbol{x}$,

$$Df(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}(-x_1 + x_1^3) & \frac{\partial}{\partial x_2}(-x_1 + x_1^3) \\ \frac{\partial}{\partial x_1}(-2x_2) & \frac{\partial}{\partial x_2}(-2x_2) \end{bmatrix},$$

so we get that

$$Df(\boldsymbol{x}) = \begin{bmatrix} -1+3\,x_1^2 & 0 \\ 0 & -2 \end{bmatrix}.$$

We only need to evaluate this matrix Df at the critical points. We start with $\boldsymbol{x}_0$,

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad Df_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The Jacobian at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is the same, so we get the same linearization at these points,

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad Df_1 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad Df_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

◁

Critical points of nonlinear systems are classified according to the eigenvalues of their corresponding linearization.

Definition 6.2.7. *A critical point $\boldsymbol{x}_0$ of a two-dimensional system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is:*

(a) an ***attractor*** *(or sink), iff both eigenvalues of Df_0 have negative real part;*
(b) a ***repeller*** *(or source), iff both eigenvalues of Df_0 have positive real part;*
(c) a ***saddle****, iff one eigenvalue of Df_0 is positive and the other is negative;*
(d) a ***center****, iff both eigenvalues of Df_0 are pure imaginary;*
(e) ***higher order*** *critical point iff at least one eigenvalue of Df_0 is zero.*

A critical point $\boldsymbol{x}_0$ is called ***hyperbolic*** *iff it belongs to cases (a-c), that is, the real part of all eigenvalues of Df_0 are nonzero.*

Example 6.2.6. Classify all the critical points of the nonlinear system

$$\begin{aligned} x_1' &= -x_1 + (x_1)^3 \\ x_2' &= -2\,x_2. \end{aligned}$$

Solution: We already know that this system has three critical points,

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

We have already computed the linearizations at these critical points too.

$$Df_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad Df_1 = Df_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

We now need to compute the eigenvalues of the Jacobian matrices above. For the critical point $\boldsymbol{x}_0$ we have $\lambda_+ = -1$, $\lambda_- = -2$, so $\boldsymbol{x}_0$ is an attractor. For the critical points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ we have $\lambda_+ = 2$, $\lambda_- = -2$, so $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are saddle points. ◁

6.2.4. The Stability of Nonlinear Systems. Sometimes the stability of two-dimensional *nonlinear* systems at a critical point is determined by the stability of the linearization at that critical point. This happens when the critical point of the linearization is hyperbolic, that is, the Jacobian matrix has eigenvalues with nonzero real part. We summarize this result in the following statement.

Theorem 6.2.8 (Hartman-Grobman). *Consider a two-dimensional nonlinear autonomous system with a continuously differentiable field* $\boldsymbol{f}$*,*

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}),$$

and consider its linearization at a hyperbolic critical point $\boldsymbol{x}_0$*,*

$$\boldsymbol{u}' = (Df_0)\,\boldsymbol{u}.$$

Then, there is a neighborhood of the hyperbolic critical point $\boldsymbol{x}_0$ *where all the solutions of the linear system can be transformed into solutions of the nonlinear system by a continuous, invertible, transformation.*

Remark: The Hartman-Grobman theorem implies that the phase portrait of the linear system in a neighborhood of a hyperbolic critical point can be transformed into the phase portrait of the nonlinear system by a continuous, invertible, transformation. When that happens we say that the two phase portraits are *topologically equivalent*.

Remark: This theorem says that, for hyperbolic critical points, the phase portrait of the linearization at the critical point is enough to determine the phase portrait of the nonlinear system near that critical point.

Example 6.2.7. Use the Hartman-Grobman theorem to sketch the phase portrait of

$$\begin{aligned} x_1' &= -x_1 + (x_1)^3 \\ x_2' &= -2\,x_2. \end{aligned}$$

Solution: We have found before that the critical points are

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

where $\boldsymbol{x}_0$ is an attractor and $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are saddle points.

The phase portrait of the linearized systems at the critical points is given in Fig 6.2.4. These critical points have all linearizations with eigenvalues having nonzero real parts. This means that the critical points are hyperbolic, so we can use the Hartman-Grobman theorem. This theorem says that the phase portrait in Fig. 6.2.4 is precisely the phase portrait of the nonlinear system in this example.

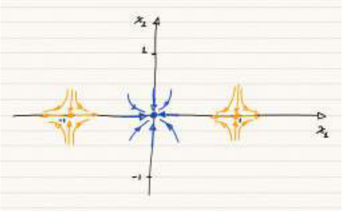


FIGURE 14. Phase portraits of the linear systems at $\boldsymbol{x}_0$, $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$.

Since we now know that Fig 6.2.4 is also the phase portrait of the nonlinear, we only need to fill in the gaps in that phase portrait. In this example, a decoupled system, we can complete the phase portrait from the symmetries of the solution. Indeed, in the x_2 direction all trajectories must decay to exponentially to the $x_2 = 0$ line. In the x_1 direction, all trajectories are attracted to $x_1 = 0$ and repelled from $x_1 = \pm 1$. The vertical lines $x_1 = 0$ and $x_1 = \pm 1$ are invariant, since $x_1' = 0$ on these lines; hence any trajectory that start on these lines stays on these lines. Similarly, $x_2 = 0$ is an invariant horizontal line. We also note that the phase portrait must be symmetric in both x_1 and x_2 axes, since the equations are invariant under the transformations $x_1 \to -x_1$ and $x_2 \to -x_2$. Putting all this extra information together we arrive to the phase portrait in Fig. 15.

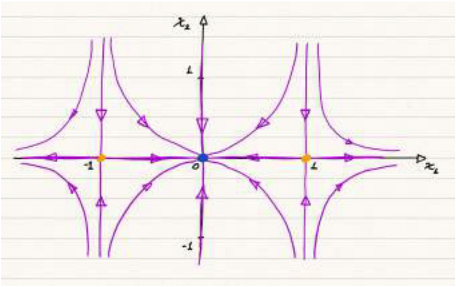


FIGURE 15. Phase portraits of the nonlinear systems in the Example 6.2.7

◁

6.2.5. Competing Species.

6.2.5. Competing Species. Suppose we have two species competing for the same food resources. Can we predict what will happen to the species population over time? Is there an equilibrium situation where both species cohabit together? Or one of the species must become extinct? If this is the case, which one?

We study in this section a particular competing species system, taken from Strogatz [12],

$$x_1' = x_1\,(3 - x_1 - 2\,x_2), \tag{6.2.1}$$

$$x_2' = x_2\,(2 - x_2 - x_1), \tag{6.2.2}$$

where $x_1(t)$ is the population of one of the species, say rabbits, and $x_2(t)$ is the population of the other species, say sheeps, at the time t. We restrict to nonnegative functions x_1, x_2.

We start finding all the critical points of the rabbits-sheeps system. We need to find all constants x_1, x_2 solutions of

$$x_1\,(3 - x_1 - 2\,x_2) = 0, \tag{6.2.3}$$

$$x_2\,(2 - x_2 - x_1) = 0. \tag{6.2.4}$$

From Eq. (6.2.3) we get that one solution is $x_1 = 0$. In that case Eq. (6.2.4) says that

$$x_2\,(2 - x_2) = 0 \quad \Rightarrow \quad x_2 = 0 \quad \text{or} \quad x_2 = 2.$$

So we got two critical points, $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. We now consider the case that $x_1 \neq 0$. In this case Eq. (6.2.3) implies

$$(3 - x_1 - 2\,x_2) = 0 \quad \Rightarrow \quad x_1 = 3 - 2\,x_2.$$

Using this equation in Eq. (6.2.4) we get that

$$x_2(2 - x_2 - 3 + 2x_2) = 0 \quad \Rightarrow \quad x_2(-1 + x_2) = 0 \quad \Rightarrow \quad \begin{cases} x_2 = 0, & \text{hence} \quad x_1 = 3, \\ \quad \text{or} & \\ x_2 = 1, & \text{hence} \quad x_1 = 1. \end{cases}$$

So we got two more critical points, $\boldsymbol{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We now proceed to find the linearization of the rabbits-sheeps system in Eqs.(6.2.1)-(6.2.2). We first compute the derivative of the field $\boldsymbol{f}$, where

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_1\,(3 - x_1 - 2\,x_2) \\ x_2\,(2 - x_2 - x_1) \end{bmatrix}.$$

The derivative of $\boldsymbol{f}$ at an arbitrary point $\boldsymbol{x}$ is

$$Df(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} (3 - 2\,x_1 - 2\,x_2) & -2\,x_1 \\ -x_2 & (2 - x_1 - 2\,x_2) \end{bmatrix}.$$

We now evaluate the matrix $Df(\boldsymbol{x})$ at each of the critical points we found.

$$\text{At} \quad \boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{we get} \quad (Df_0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

This coefficient matrix has eigenvalues $\lambda_{0+} = 3$ and $\lambda_{0-} = 2$, both positive, which means that the critical point $\boldsymbol{x}_0$ is a repeller. To sketch the phase portrait we will need the corresponding eigenvectors, $\boldsymbol{v}_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\text{At} \quad \boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \text{we get} \quad (Df_1) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}.$$

This coefficient matrix has eigenvalues $\lambda_{1+} = -1$ and $\lambda_{1-} = -2$, both negative, which means that the critical point $\boldsymbol{x}_1$ is an attractor. One can check that the corresponding eigenvectors are $\boldsymbol{v}_1^+ = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{v}_1^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\text{At} \quad \boldsymbol{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{we get} \quad (Df_2) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}.$$

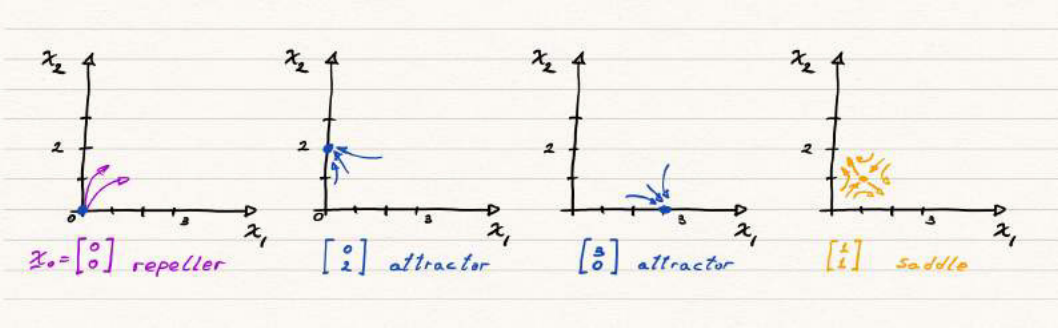
This coefficient matrix has eigenvalues $\lambda_{2+} = -1$ and $\lambda_{2-} = -3$, both negative, which means that the critical point $\boldsymbol{x}_2$ is an attractor. One can check that the corresponding eigenvectors are $\boldsymbol{v}_2^+ = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2^- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\text{At} \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{we get} \quad (Df_3) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}.$$

One can check that this coefficient matrix has eigenvalues $\lambda_{3+} = -1+\sqrt{2}$ and $\lambda_{3-} = -1-\sqrt{2}$, which means that the critical point $\boldsymbol{x}_3$ is a saddle. One can check that the corresponding eigenvectors are $\boldsymbol{v}_3^+ = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_3^- = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$. We summarize this information about the linearized systems in the following picture.

We would like to have the complete phase portrait for the nonlinear system, that is, we would like to fill the gaps in Fig. 17. This is difficult to do analytically in this example as well as in general nonlinear autonomous systems. At this point is where we need to turn to computer generated solutions to fill the gaps in Fig. 17. The result is in Fig. 18.

We can now study the phase portrait in Fig. 18 to obtain some biological insight on the rabbits-sheeps system. The picture on the right says that most of the time one species drives the other to extinction. If the initial data for the system is a point on the blue region, called the rabbit basin, then the solution evolves in time toward the critical point $\boldsymbol{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. This means that the sheep become extinct. If the initial data for the system is a point on the green region, called the sheep basin, then the solution evolves in time toward the critical point $\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. This means that the rabbits become extinct.



x_2
2
0
3
x_1
$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ repeller
$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ attractor
$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ attractor
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ saddle

FIGURE 16. The linearizations of the rabbits-sheeps system in Eqs. (6.2.1)-(6.2.2).

We now notice that all these critical points have nonzero real part, that means they are hyperbolic critical points. Then we can use Hartman-Grobman Theorem 6.2.8 to construct the phase portrait of the nonlinear system in (6.2.1)-(6.2.2) around these critical points. The Hartman-Grobman theorem says that the qualitative structure of the phase portrait for the linearized system is the same for the phase portrait of the nonlinear system around the critical point. So we get the picture in Fig. 17.

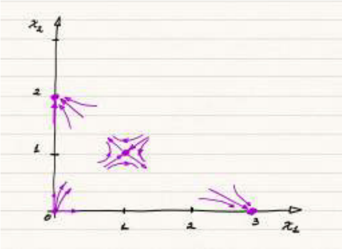


FIGURE 17. Phase Portrait for Eqs. (6.2.1)-(6.2.2).

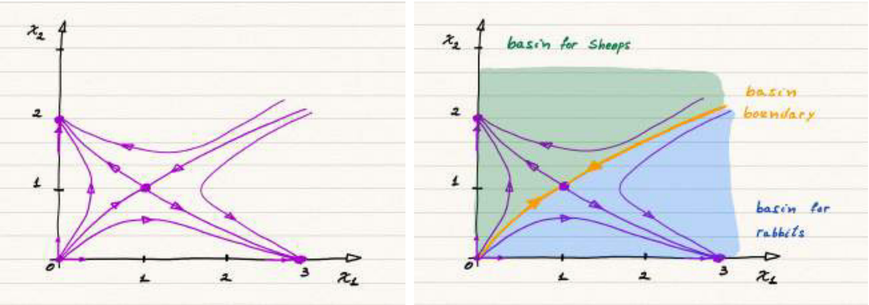


FIGURE 18. The phase portrait of the rabbits-sheeps system in Eqs. (6.2.1)-(6.2.2).

The two basins of attractions are separated by a curve, called the basin boundary. Only when the initial data lies on that curve the rabbits and sheeps coexist with neither becoming extinct. The solution moves towards the critical point $\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, the populations of rabbits and sheep become equal to each other as time goes to infinity. But, if we pick an initial data outside this basin boundary, no matter how close this boundary, one of the species becomes extinct.

6.2.6. Exercises.

6.2.1.- * Consider the autonomous system

$$x' = x(1 - x - y)$$

$$y' = y\Big(\frac{3}{4} - y - \frac{1}{2}x\Big)$$

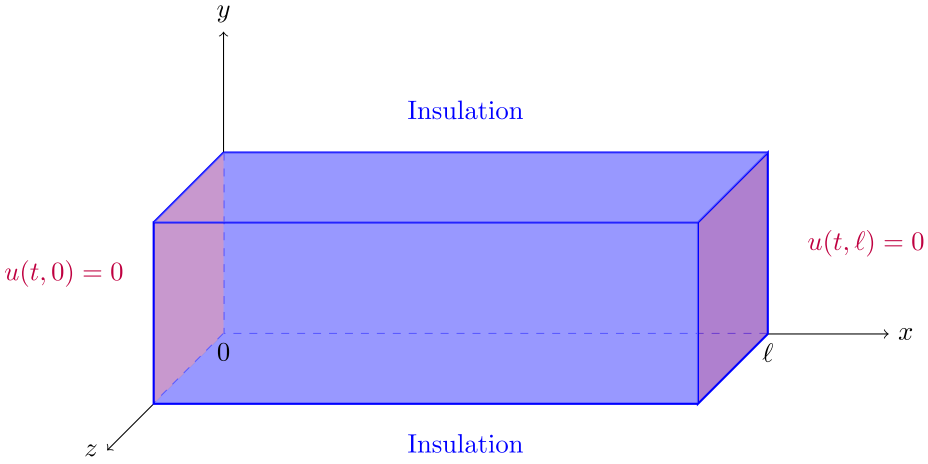
(1) Find the critical points of the system above.
(2) Find the linearization at each of the critical points above.
(3) Classify the critical points above as attractors, repellers, or saddle points.
(4) Sketch a qualitative phase portrait of the solutions of the system above.

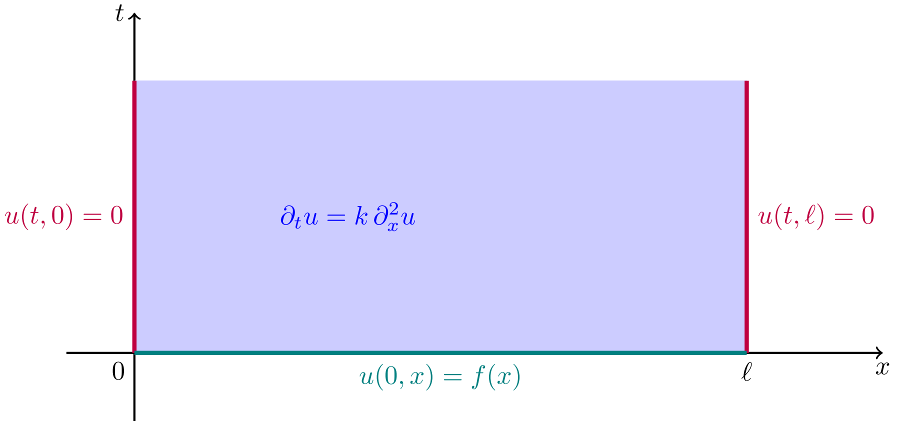
6.2.2.- .

CHAPTER 7

Boundary Value Problems

We study the a simple case of the Sturm-Liouville Problem, we then present how to compute the Fourier series expansion of continuous and discontinuous functions. We end this chapter introducing the separation of variables method to find solutions of a partial differential equation, the heat equation.





7.1. Eigenfunction Problems

In this Section we consider second order, linear, ordinary differential equations. In the first half of the Section we study boundary value problems for these equations and in the second half we focus on a particular type of boundary value problems, called the eigenvalue-eigenfunction problem for these equations.

7.1.1. Two-Point Boundary Value Problems.

We start with the definition of a two-point boundary value problem.

Definition 7.1.1. *A **two-point boundary value problem** (BVP) is the following: Find solutions to the differential equation*

$$y'' + a_1(x)\, y' + a_0(x)\, y = b(x)$$

satisfying the boundary conditions (BC)

$$\begin{aligned} b_1\, y(x_1) + b_2\, y'(x_1) &= y_1,\\ \tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) &= y_2, \end{aligned}$$

*where b_1, b_2, $\tilde{b}_1$, $\tilde{b}_2$, x_1, x_2, y_1, and y_2 are given and $x_1 \neq x_2$. The boundary conditions are **homogeneous** iff $y_1 = 0$ and $y_2 = 0$*

Remarks:

(a) The two boundary conditions are held at *different* points, $x_1 \neq x_2$.
(b) Both y and y' may appear in the boundary condition.

Example 7.1.1. We now show four examples of boundary value problems that differ only on the boundary conditions: Solve the different equation

$$y'' + a_1\, y' + a_0\, y = e^{-2t}$$

with the boundary conditions at $x_1 = 0$ and $x_2 = 1$ given below.

(a)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 1, & b_2 = 0, \\ \tilde{b}_1 = 1, & \tilde{b}_2 = 0. \end{Bmatrix}$$

(b)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 1, & b_2 = 0, \\ \tilde{b}_1 = 0, & \tilde{b}_2 = 1. \end{Bmatrix}$$

(c)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 0, & b_2 = 1, \\ \tilde{b}_1 = 1, & \tilde{b}_2 = 0. \end{Bmatrix}$$

(d)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 0, & b_2 = 1, \\ \tilde{b}_1 = 0, & \tilde{b}_2 = 1. \end{Bmatrix}$$

(e)

$$\text{BC:} \quad \begin{Bmatrix} 2\, y(0) + y'(0) = y_1, \\ y'(1) + 3\, y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = 2, & b_2 = 1, \\ \tilde{b}_1 = 1, & \tilde{b}_2 = 3. \end{Bmatrix}$$

◁

7.1.2. Comparison: IVP and BVP. We now review the initial boundary value problem for the equation above, which was discussed in Sect. 2.1, where we showed in Theorem 2.1.2 that this initial value problem always has a unique solution.

Definition 7.1.2 (IVP)**.** *Find all solutions of the differential equation* $y'' + a_1\, y' + a_0\, y = 0$ *satisfying the initial condition (IC)*

$$y(t_0) = y_0, \qquad y'(t_0) = y_1. \tag{7.1.1}$$

Remarks: In an initial value problem we usually the following happens.

- The variable t represents time.
- The variable y represents position.
- The IC are position and velocity at the initial time.

A typical boundary value problem that appears in many applications is the following.

Definition 7.1.3 (BVP)**.** *Find all solutions of the differential equation* $y'' + a_1\, y' + a_0\, y = 0$ *satisfying the boundary condition (BC)*

$$y(0) = y_0, \qquad y(L) = y_1, \qquad L \neq 0. \tag{7.1.2}$$

Remarks: In a boundary value problem we usually the following happens.

- The variable x represents position.
- The variable y may represents a physical quantity such us temperature.
- The BC are the temperature at two different positions.

The names "initial value problem" and "boundary value problem" come from physics. An example of the former is to solve Newton's equations of motion for the position function of a point particle that starts at a given initial position and velocity. An example of the latter is to find the equilibrium temperature of a cylindrical bar with thermal insulation on the round surface and held at constant temperatures at the top and bottom sides.

Let's recall an important result we saw in § 2.1 about solutions to initial value problems.

Theorem 7.1.4 (IVP)**.** *The equation* $y'' + a_1\, y' + a_0\, y = 0$ *with IC* $y(t_0) = y_0$ *and* $y'(t_0) = y_1$ *has a unique solution* y *for each choice of the IC.*

The solutions to boundary value problems are more complicated to describe. A boundary value problem may have a unique solution, or may have infinitely many solutions, or may have no solution, depending on the boundary conditions. In the case of the boundary value problem in Def. 7.1.3 we get the following.

Theorem 7.1.5 (BVP)**.** *The equation* $y'' + a_1\, y' + a_0\, y = 0$ *with BC* $y(0) = y_0$ *and* $y(L) = y_1$*, wilt* $L \neq 0$ *and with* $r_\pm$ *roots of the characteristic polynomial* $p(r) = r^2 + a_1 r + a_0$*, satisfy the following.*

(A) If $r_+ \neq r_-$ *are reals, then the BVP above has a unique solution for all* $y_0,\, y_1 \in \mathbb{R}$*.*

(B) If $r_\pm = \alpha \pm i\,\beta$ *are complex, with* $\alpha,\, \beta \in \mathbb{R}$*, then the solution of the BVP above belongs to one of the following three possibilities:*

  (i) There exists a unique solution;

  (ii) There exists infinitely many solutions;

(iii) There exists no solution.

Proof of Theorem 7.1.5:

Part (A): If $r_+ \neq r_-$ are reals, then the general solution of the differential equation is

$$y(x) = c_+ \, e^{r_+ x} + c_- \, e^{r_- x}.$$

The boundary conditions are

$$\left.\begin{aligned} y_0 &= y(0) = c_+ + c_- \\ y_1 &= y(L) = c_+ \, e^{c_+ L} + c_- \, e^{c_- L} \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

This system for c_+, c_- has a unique solution iff the coefficient matrix is invertible. But its determinant is

$$\begin{vmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{vmatrix} = e^{r_- L} - e^{r_+ L}.$$

Therefore, if the roots $r_+ \neq r_-$ are reals, then $e^{r_- L} \neq e^{r_+ L}$, hence there is a unique solution c_+, c_-, which in turn fixes a unique solution y of the BVP.

In the case that $r_+ = r_- = r_0$, then we have to start over, since the general solution of the differential equation is

$$y(x) = (c_1 + c_2 \, x) \, e^{r_0 x}, \qquad c_1, c_2 \in \mathbb{R}.$$

Again, the boundary conditions in Eq. (7.1.2) determine the values of the constants c_1 and c_2 as follows:

$$\left.\begin{aligned} y_0 &= y(0) = c_1 \\ y_1 &= y(L) = c_1 e^{r_0 L} + c_2 L \, e^{r_0 L} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ e^{r_0 L} & L \, e^{r_0 L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

This system for c_1, c_2 has a unique solution iff the coefficient matrix is invertible. But its determinant is

$$\begin{vmatrix} 1 & 0 \\ e^{r_0 L} & L \, e^{r_0 L} \end{vmatrix} = L \, e^{r_0 L}$$

So, for $L \neq 0$ the determinant above is nonzero, then there is a unique solution c_1, c_2, which in turn fixes a unique solution y of the BVP.

Part (B): If $r_\pm = \alpha \pm i\beta$, that is complex, then

$$e^{r_\pm L} = e^{(\alpha \pm i\beta)L} = e^{\alpha L}(\cos(\beta L) \pm i \sin(\beta L)),$$

therefore

$$\begin{aligned} e^{r_- L} - e^{r_+ L} &= e^{\alpha L}\big(\cos(\beta L) - i \sin(\beta L) - \cos(\beta L) - i \sin(\beta L)\big) \\ &= -2i \, e^{\alpha L} \, \sin(\beta L). \end{aligned}$$

We conclude that

$$e^{r_- L} - e^{r_+ L} = -2i \, e^{\alpha L} \, \sin(\beta L) = 0 \quad \Leftrightarrow \quad \beta L = n\pi.$$

So for $\beta L \neq n\pi$ the BVP has a unique solution, case (Bi). But for $\beta L = n\pi$ the BVP has either no solution or infinitely many solutions, cases (Bii) and (Biii). This establishes the Theorem. ☐

Example 7.1.2. Find all solutions to the BVPs $y'' + y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = 0. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = -1. \end{cases}$$

Solution: We first find the roots of the characteristic polynomial $r^2 + 1 = 0$, that is, $r_\pm = \pm i$. So the general solution of the differential equation is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$0 = y(\pi) = -c_1 \quad \Rightarrow \quad c_1 = 0.$$

Therefore, there is no solution.

BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_2 = 1.$$

So there is a unique solution $y(x) = \cos(x) + \sin(x)$.

BC (c):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi) = -c_1 \quad \Rightarrow \quad c_2 = 1.$$

Therefore, c_2 is arbitrary, so we have infinitely many solutions

$$y(x) = \cos(x) + c_2 \sin(x), \qquad c_2 \in \mathbb{R}.$$

◁

Example 7.1.3. Find all solutions to the BVPs $y'' + 4\,y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi/4) = -1. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = -1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases}$$

Solution: We first find the roots of the characteristic polynomial $r^2 + 4 = 0$, that is, $r_\pm = \pm 2i$. So the general solution of the differential equation is

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

BC (a):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi/4) = c_2 \quad \Rightarrow \quad c_2 = -1.$$

Therefore, there is a unique solution $y(x) = \cos(2x) - \sin(2x)$.

BC (b):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$-1 = y(\pi/2) = -c_1 \quad \Rightarrow \quad c_1 = 1.$$

So, c_2 is arbitrary and we have infinitely many solutions

$$y(x) = \cos(2x) + c_2 \sin(2x), \qquad c_2 \in \mathbb{R}.$$

BC (c):

$$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1.$$

$$1 = y(\pi/2) = -c_1 \quad \Rightarrow \quad c_2 = -1.$$

Therefore, we have no solution. ◁

7.1.3. Eigenfunction Problems. We now focus on boundary value problems that have infinitely many solutions. A particular type of these problems are called an eigenfunction problems. They are similar to the eigenvector problems we studied in § 8.3. Recall that the eigenvector problem is the following: Given an $n \times n$ matrix A, find all numbers λ and nonzero vectors $\boldsymbol{v}$ solution of the algebraic linear system

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

We saw that for each λ there are infinitely many solutions $\boldsymbol{v}$, because if $\boldsymbol{v}$ is a solution so is any multiple $a\boldsymbol{v}$. An eigenfunction problem is something similar.

Definition 7.1.6. *An* ***eigenfunction problem*** *is the following: Given a linear operator* $L(y) = y'' + a_1\, y' + a_0\, y$, *find a number* λ *and a nonzero function* y *solution of*

$$L(y) = -\lambda y,$$

with homogeneous boundary conditions

$$b_1\, y(x_1) + b_2\, y'(x_1) = 0,$$
$$\tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) = 0.$$

Remarks:

- Notice that $y = 0$ is always a solution of the BVP above.
- Eigenfunctions are the nonzero solutions of the BVP above.
- Hence, the eigenfunction problem is a BVP with infinitely many solutions.
- So, we look for λ such that the operator $L(y) + \lambda\, y$ has characteristic polynomial with complex roots.
- So, λ is such that $L(y) + \lambda\, y$ has oscillatory solutions.
- Our examples focus on the linear operator $L(y) = y''$.

Example 7.1.4. Find all numbers λ and nonzero functions y solutions of the BVP

$$y'' + \lambda\, y = 0, \quad \text{with} \quad y(0) = 0, \quad y(L) = 0, \qquad L > 0.$$

Solution: We divide the problem in three cases: **(a)** $\lambda < 0$, **(b)** $\lambda = 0$, and **(c)** $\lambda > 0$.
Case (a): $\lambda = -\mu^2 < 0$, so the equation is $y'' - \mu^2 y = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is $y = c_+\, e^{\mu x} + c_-\, e^{-\mu x}$. The BC imply

$$0 = y(0) = c_+ + c_-, \qquad 0 = y(L) = c_+\, e^{\mu L} + c_-\, e^{-\mu L}.$$

So from the first equation we get $c_+ = -c_-$, so

$$0 = -c_-\, e^{\mu L} + c_-\, e^{-\mu L} \quad \Rightarrow \quad -c_-(e^{\mu L} - e^{-\mu L}) = 0 \quad \Rightarrow \quad c_- = 0, \quad c_+ = 0.$$

So the only the solution is $y = 0$, then there are no eigenfunctions with negative eigenvalues.
Case (b): $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y = c_0 + c_1 x.$$

The BC imply

$$0 = y(0) = c_0, \qquad 0 = y(L) = c_1 L \quad \Rightarrow \quad c_1 = 0.$$

So the only solution is $y = 0$, then there are no eigenfunctions with eigenvalue $\lambda = 0$.
Case (c): $\lambda = \mu^2 > 0$, so the equation is $y'' + \mu^2 y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is $y = c_+ \cos(\mu x) + c_- \sin(\mu x)$. The BC imply

$$0 = y(0) = c_+, \qquad 0 = y(L) = c_+ \cos(\mu L) + c_- \sin(\mu L).$$

Since $c_+ = 0$, the second equation above is

$$c_- \sin(\mu L) = 0, \qquad c_- \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi.$$

So we get $\mu_n = n\pi/L$, hence the eigenvalue eigenfunction pairs are

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = c_n \sin\Big(\frac{n\pi x}{L}\Big).$$

Since we need only one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = \sin\Big(\frac{n\pi x}{L}\Big), \qquad n \geqslant 1.$$

◁

Example 7.1.5. Find the numbers λ and the nonzero functions y solutions of the BVP

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y'(L) = 0, \qquad L > 0.$$

Solution: We divide the problem in three cases: **(a)** $\lambda < 0$, **(b)** $\lambda = 0$, and **(c)** $\lambda > 0$.
Case (a): Let $\lambda = -\mu^2$, with $\mu > 0$, so the equation is $y'' - \mu^2 y = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu,$$

The general solution is $y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}$. The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c_1 + c_2, \\ 0 &= y'(L) = -\mu c_1 e^{-\mu L} + \mu c_2 e^{\mu L} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ -\mu e^{-\mu L} & \mu e^{\mu L} \end{vmatrix} = \mu\big(e^{\mu L} + e^{-\mu L}\big) \neq 0.$$

So, the linear system above for c_1, c_2 has a unique solution $c_1 = c_2 = 0$. Hence, we get the only solution $y = 0$. This means there are no eigenfunctions with negative eigenvalues.
Case (b): Let $\lambda = 0$, so the differential equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$0 = y(0) = c_1, \qquad 0 = y'(L) = c_2.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 0$.
Case (c): Let $\lambda = \mu^2$, with $\mu > 0$, so the equation is $y'' + \mu^2 y = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu\, i.$$

The general solution is $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. The BC imply

$$\left.\begin{aligned} 0 &= y(0) = c_1, \\ 0 &= y'(L) = -\mu c_1 \sin(\mu L) + \mu c_2 \cos(\mu L) \end{aligned}\right\} \quad \Rightarrow \quad c_2 \cos(\mu L) = 0.$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\cos(\mu L) = 0 \quad \Leftrightarrow \quad \mu_n L = (2n-1)\frac{\pi}{2}, \qquad n \geqslant 1.$$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = -\frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = c_n \sin\Big(\frac{(2n-1)\pi x}{2L}\Big), \qquad n \geqslant 1.$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = -\frac{(2n-1)^2\pi^2}{4L^2}, \qquad y_n(x) = \sin\Big(\frac{(2n-1)\pi x}{2L}\Big), \qquad n \geqslant 1.$$

◁

Example 7.1.6. Find the numbers λ and the nonzero functions y solutions of the BVP

$$x^2\, y'' - x\, y' = -\lambda\, y, \qquad y(1) = 0, \quad y(\ell) = 0, \quad \ell > 1.$$

Solution: Let us rewrite the equation as

$$x^2\, y'' - x\, y' + \lambda y = 0.$$

This is an Euler equidimensional equation. From § 2.4 we know we need to look for the solutions $r_\pm$ of the indicial polynomial

$$r(r-1) - r + \lambda = 0 \quad \Rightarrow \quad r^2 - 2r + \lambda = 0 \quad \Rightarrow \quad r_\pm = 1 \pm \sqrt{1-\lambda}.$$

Case (a): Let $1 - \lambda = 0$, so we have a repeated root $r_+ = r_- = 1$. The general solution to the differential equation is

$$y(x) = \big(c_1 + c_2 \ln(x)\big)\, x.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1, \\ 0 &= y(\ell) = \big(c_1 + c_2 \ln(\ell)\big)\, \ell \end{aligned}\right\} \quad \Rightarrow \quad c_2 \ell \ln(\ell) = 0 \quad \Rightarrow \quad c_2 = 0.$$

So the only solution is $y = 0$. This means there are no eigenfunctions with eigenvalue $\lambda = 1$.

Case (b): Let $1 - \lambda > 0$, so we can rewrite it as $1 - \lambda = \mu^2$, with $\mu > 0$. Then, $r_\pm = 1 \pm \mu$, and so the general solution to the differential equation is given by

$$y(x) = c_1 x^{(1-\mu)} + c_2 x^{(1+\mu)},$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2, \\ 0 &= y(\ell) = c_1 \ell^{(1-\mu)} + c_2 \ell^{(1+\mu)} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The matrix above is invertible, because

$$\begin{vmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{vmatrix} = \ell\, \big(\ell^{\mu} - \ell^{-\mu}\big) \neq 0 \quad \Leftrightarrow \quad \ell \neq \pm 1.$$

Since $\ell > 1$, the matrix above is invertible, and the linear system for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$. Hence we get the only solution $y = 0$. This means there are no eigenfunctions with eigenvalues $\lambda < 1$.

Case (c): Let $1 - \lambda < 0$, so we can rewrite it as $1 - \lambda = -\mu^2$, with $\mu > 0$. Then $r_\pm = 1 \pm i\mu$, and so the general solution to the differential equation is

$$y(x) = x \big[c_1 \cos\big(\mu \ln(x)\big) + c_2 \sin\big(\mu \ln(x)\big)\big].$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1, \\ 0 &= y(\ell) = c_1 \ell \cos\big(\mu \ln(\ell)\big) + c_2 \ell \sin\big((\mu \ln(\ell)\big) \end{aligned}\right\} \quad \Rightarrow \quad c_2 \ell \sin\big(\mu \ln(\ell)\big) = 0.$$

Since we are interested in nonzero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\sin\big(\mu \ln(\ell)\big) = 0 \quad \Leftrightarrow \quad \mu_n \ln(\ell) = n\pi, \qquad n \geqslant 1.$$

Recalling that $1 - \lambda_n = -\mu_n^2$, we get $\lambda_n = 1 + \mu_n^2$, hence,

$$\lambda_n = 1 + \frac{n^2\pi^2}{\ln^2(\ell)}, \qquad y_n(x) = c_n x \sin\Big(\frac{n\pi \ln(x)}{\ln(\ell)}\Big), \qquad n \geqslant 1.$$

Since we only need one eigenfunction for each eigenvalue, we choose $c_n = 1$, and we get

$$\lambda_n = 1 + \frac{n^2\pi^2}{\ln^2(\ell)}, \qquad y_n(x) = x \sin\Big(\frac{n\pi \ln(x)}{\ln(\ell)}\Big), \qquad n \geqslant 1.$$

◁

7.1.4. Exercises.

7.1.1.- .

7.1.2.- .

7.2. Overview of Fourier series

A vector in three dimensional space can be decomposed as a linear combination of its components in a vector basis. If the basis vectors are all mutually perpendicular—an orthogonal basis—then there is a simple formula for the vector components. This is the Fourier expansion theorem for vectors in space. In this section we generalize these ideas from the three dimensional space to the infinite dimensional space of continuous functions. We introduce a notion of orthogonality among functions and we choose a particular orthogonal basis in this space. Then we state that any continuous function can be decomposed as a linear combination of its components in that orthogonal basis. This is the Fourier series expansion theorem for continuous functions.

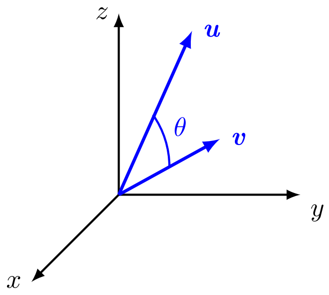
7.2.1. Fourier Expansion of Vectors.

We review the basic concepts about vectors in $\mathbb{R}^3$ we will need to generalize to the space of functions. These concepts include: the dot (or inner) product of two vectors, orthogonal and orthonormal set of vectors, Fourier expansion (or orthonormal expansion) of vectors, and vector approximations.

Definition 7.2.1. *The* ***dot product*** *of two vectors* $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^3$ *is*

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}|\,|\boldsymbol{v}|\,\cos(\theta),$$

with $|\boldsymbol{u}|$, $|\boldsymbol{v}|$ *the magnitude of the vectors, and* $\theta \in [0, \pi]$ *the angle in between them.*



The magnitude of a vector $\boldsymbol{u}$ can be written as

$$|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}.$$

A vector $\boldsymbol{u}$ is a unit vector iff

$$\boldsymbol{u} \cdot \boldsymbol{u} = 1.$$

Any vector $\boldsymbol{v}$ can be rescaled into a unit vector by dividing by its magnitude. So, the vector $\boldsymbol{u}$ below is a unit vector,

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}.$$

The dot product tries to capture the notion of projection of one vector onto another. In the case that one of the vectors is a unit vector, the dot product is exactly the projection of the second vector onto the unit vector,

$$\boldsymbol{v} \cdot \boldsymbol{u} = |\boldsymbol{v}| \cos(\theta), \qquad \text{for} \qquad |\boldsymbol{u}| = 1.$$

The dot product above satisfies the following properties.

Theorem 7.2.2. *For every* $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^3$ *and every* a, $b \in \mathbb{R}$ *holds,*

(a) Positivity: $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ *iff* $\boldsymbol{u} = 0$*; and* $\boldsymbol{u} \cdot \boldsymbol{u} > 0$ *for* $cb\boldsymbol{u} \neq 0$.
(b) Symmetry: $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.
(c) Linearity: $(a\boldsymbol{u} + b\boldsymbol{v}) \cdot \boldsymbol{w} = a\,(\boldsymbol{u} \cdot \boldsymbol{w}) + b\,(\boldsymbol{v} \cdot \boldsymbol{w})$.

When two vectors are perpendicular—no projection of one onto the other—their dot product vanishes.

Theorem 7.2.3. *The vectors* $\boldsymbol{u}$, $\boldsymbol{v}$ *are orthogonal (perpendicular) iff* $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

A set of vectors is an *orthogonal set* if all the vectors in the set are mutually perpendicular. An *orthonormal set* is an orthogonal set where all the vectors are unit vectors.

Example 7.2.1. The set of vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ used in physics is an orthonormal set in $\mathbb{R}^3$.

Solution: These are the vectors

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

As we can see in Fig. 1, they are mutually perpendicular, and have unit magnitude, that is,

$$\begin{aligned} \boldsymbol{i}\cdot\boldsymbol{j} &= 0, & \boldsymbol{i}\cdot\boldsymbol{i} &= 1, \\ \boldsymbol{i}\cdot\boldsymbol{k} &= 0, & \boldsymbol{j}\cdot\boldsymbol{j} &= 1, \\ \boldsymbol{j}\cdot\boldsymbol{k} &= 0. & \boldsymbol{k}\cdot\boldsymbol{k} &= 1. \end{aligned}$$

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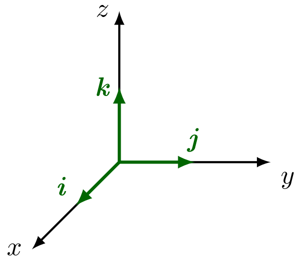

FIGURE 1. Vectors $\boldsymbol{i}$,$\boldsymbol{j}$,$\boldsymbol{k}$.

The Fourier expansion theorem says that the set above is not just a set, it is a basis—any vector in $\mathbb{R}^3$ can be decomposed as a linear combination of the basis vectors. Furthermore, there is a simple formula for the vector components.

Theorem 7.2.4. *The orthonormal set* $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ *is an orthonormal basis, that is, every vector* $\boldsymbol{v} \in \mathbb{R}^3$ *can be decomposed as*

$$\boldsymbol{v} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k}.$$

The orthonormality of the vector set implies a formula for the vector components

$$v_x = \boldsymbol{v}\cdot\boldsymbol{i}, \qquad v_y = \boldsymbol{v}\cdot\boldsymbol{j}, \qquad v_z = \boldsymbol{v}\cdot\boldsymbol{k}.$$

The vector components are the dot product of the whole vector with each basis vector. The decomposition above allow us to introduce vector approximations.

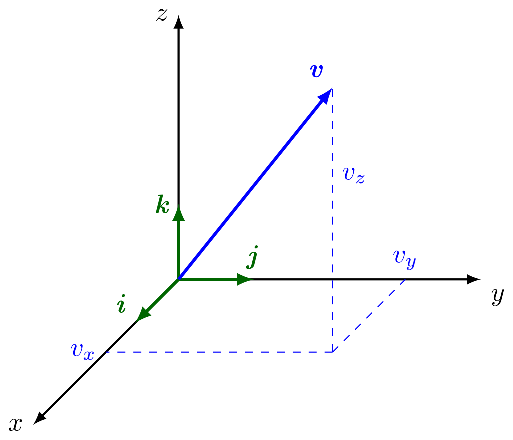

FIGURE 2. Fourier expansion of a vector in an orthonormal basis.

The Fourier expansion of a vector in an orthonormal basis allows us to introduce vector approximations. We just cut the Fourier expansion at the first, second, or third term:

$$\begin{aligned} \boldsymbol{v}^{(1)} &= v_x\,\boldsymbol{i}, \\ \boldsymbol{v}^{(2)} &= v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j}, \\ \boldsymbol{v}^{(3)} &= v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k}. \end{aligned}$$

Such vector approximations are silly to do in three dimensional space. But they can be useful if we work in a large dimensional space. And they become an essential tool when we work in an infinite dimensional space, such as the space of continuous functions.

7.2.2. Fourier Expansion of Functions. The ideas described above for vectors in $\mathbb{R}^3$ can be extended to functions. We start introducing a notion of projection—hence of perpendicularity—among functions. Unlike it happens in $\mathbb{R}^3$, we now do not have a geometric intuition that can help us find such a product. So we look for any dot product of functions having the positivity property, the symmetry property, and the linearity property. Here is one product with these properties.

Definition 7.2.5. *The* ***dot product*** *of two functions f, g on $[-L, L]$ is*

$$f \cdot g = \int_{-L}^{L} f(x)\, g(x)\, dx.$$

The dot product above takes two functions and produces a number. And one can verify that the product has the following properties.

Theorem 7.2.6. *For every functions f, g, h and every a, $b \in \mathbb{R}$ holds,*

(a) Positivity: $f \cdot f = 0$ iff $f = 0$; and $f \cdot f > 0$ for $f \neq 0$.
(b) Symmetry: $f \cdot g = g \cdot f$.
(c) Linearity: $(a\,f + b\,g) \cdot h = a\,(f \cdot h) + b\,(g \cdot h)$.

Remarks: The *magnitude* of a function f is the nonnegative number

$$\|f\| = \sqrt{f \cdot f} = \Big(\int_{-L}^{L} \big(f(x)\big)^2\, dx \Big)^{1/2}.$$

We use a double bar to denote the magnitude so we do not confuse it with $|f|$, which means the absolute value. A function f is a *unit function* iff $f \cdot f = 1$.

Since we do not have a geometric intuition for perpendicular functions, we need to define such a notion on functions using the dot product. Therefore, the following statement for functions is a definition, unlike for vectors in space where it is a theorem.

Definition 7.2.7. *Two functions f, g are* ***orthogonal*** (perpendicular) *iff $f \cdot g = 0$.*

A set of functions is an *orthogonal set* if all the functions in the set are mutually perpendicular. An *orthonormal set* is an orthogonal set where all the functions are unit functions.

Theorem 7.2.8. *An orthonormal set in the space of continuous functions on $[-L, L]$ is*

$$\Big\{ \tilde{u}_0 = \frac{1}{\sqrt{2L}}, \quad \tilde{u}_n = \frac{1}{\sqrt{L}} \cos\Big(\frac{n\pi x}{L}\Big), \quad \tilde{v}_n = \frac{1}{\sqrt{L}} \sin\Big(\frac{n\pi x}{L}\Big) \Big\}_{n=1}^{\infty}.$$

Remark: To show that the set above is orthogonal we need to show that the dot product of any two different functions in the set vanishes—the three equations below on the left. To show that the set if orthonormal we also need to show that all the functions in the set are unit functions—the two equations below on the right.

$$\begin{aligned} \tilde{u}_m \cdot \tilde{u}_n &= 0, \quad m \neq n. & \qquad \tilde{u}_n \cdot \tilde{u}_n &= 1, \quad \text{for all } n. \\ \tilde{v}_m \cdot \tilde{v}_n &= 0, \quad m \neq n. & \qquad \tilde{v}_n \cdot \tilde{v}_n &= 1, \quad \text{for all } n. \\ \tilde{u}_m \cdot \tilde{v}_n &= 0, \quad \text{for all } m, n. & & \end{aligned}$$

Example 7.2.2. The normalization condition is simple to see, because for $n \geqslant 1$ holds

$$\tilde{u}_n \cdot \tilde{u}_n = \int_{-L}^{L} \frac{1}{\sqrt{L}} \cos\Big(\frac{n\pi x}{L}\Big) \frac{1}{\sqrt{L}} \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{1}{L} \int_{-L}^{L} \cos^2\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{1}{L}\, L = 1.$$

◁

The orthogonality of the set above is equivalent to the following statement about the functions sine and cosine.

Theorem 7.2.9. *The following relations hold for all n, $m \in \mathbb{N}$,*

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big) \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ L & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \sin\Big(\frac{m\pi x}{L}\Big)\, dx = 0.$$

Proof of Theorem 7.2.9: Just recall the following trigonometric identities:

$$\cos(\theta)\, \cos(\phi) = \frac{1}{2}\big[\cos(\theta + \phi) + \cos(\theta - \phi)\big],$$

$$\sin(\theta)\, \sin(\phi) = \frac{1}{2}\big[\cos(\theta - \phi) - \cos(\theta + \phi)\big],$$

$$\sin(\theta)\, \cos(\phi) = \frac{1}{2}\big[\sin(\theta + \phi) + \sin(\theta - \phi)\big].$$

So, From the trigonometric identities above we obtain

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \frac{1}{2} \int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\, dx + \frac{1}{2} \int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\, dx.$$

First, assume $n > 0$ or $m > 0$, then the first term vanishes, since

$$\frac{1}{2} \int_{-L}^{L} \cos\Big[\frac{(n+m)\pi x}{L}\Big]\, dx = \frac{L}{2(n+m)\pi}\, \sin\Big[\frac{(n+m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

Still for $n > 0$ or $m > 0$, assume that $n \neq m$, then the second term above is

$$\frac{1}{2} \int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\, dx = \frac{L}{2(n-m)\pi}\, \sin\Big[\frac{(n-m)\pi x}{L}\Big]\Big|_{-L}^{L} = 0.$$

Again, still for $n > 0$ or $m > 0$, assume that $n = m \neq 0$, then

$$\frac{1}{2} \int_{-L}^{L} \cos\Big[\frac{(n-m)\pi x}{L}\Big]\, dx = \frac{1}{2} \int_{-L}^{L} dx = L.$$

Finally, in the case that both $n = m = 0$ is simple to see that

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \int_{-L}^{L} dx = 2L.$$

The remaining equations in the Theorem are proven in a similar way. This establishes the Theorem. ☐

Remark: Instead of an orthonormal set we will use an *orthogonal* set, which is often used in the literature on Fourier series:

$$\Big\{u_0 = \frac{1}{2},\ u_n = \cos\Big(\frac{n\pi x}{L}\Big),\ v_n = \Big(\frac{n\pi x}{L}\Big)\Big\}_{n=1}^{\infty}.$$

Theorem 7.2.10 (Fourier Expansion)**.** *The orthogonal set*

$$\Big\{u_0 = \frac{1}{2},\ u_n = \cos\Big(\frac{n\pi x}{L}\Big),\ v_n = \Big(\frac{n\pi x}{L}\Big)\Big\}_{n=1}^{\infty} \tag{7.2.1}$$

is an orthogonal basis of the space of continuous functions on $[-L, L]$*, that is, any continuous function on* $[-L, L]$ *can be decomposed as*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big).$$

Moreover, the coefficients above are given by the formulas

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

Furthermore, if f *is piecewise continuous, then the function*

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big),$$

satisfies $f_F(x) = f(x)$ *for all* x *where* f *is continuous, while for all* x_0 *where* f *is discontinuous it holds*

$$f_F(x_0) = \frac{1}{2}\Big(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\Big)$$

Idea of the Proof of Theorem 7.2.10: It is not simple to prove that the set in 7.2.1 is a basis, that is every continuous function on $[-L, L]$ can be written as a linear combination

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big).$$

We skip that part of the proof. But once we have the expansion above, it is not difficult to find a formula for the coefficients a_0, a_n, and b_n, for $n \geqslant 1$. To find a coefficient b_m we just multiply the expansion above by $\sin(m\pi x/L)$ and integrate on $[-L, L]$, that is,

$$f(x)\cdot\sin\Big(\frac{m\pi x}{L}\Big) = \Big(\frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big)\Big)\cdot\sin\Big(\frac{m\pi x}{L}\Big).$$

The linearity property of the dot product implies

$$
\begin{aligned}
f(x)\cdot\sin\Big(\frac{m\pi x}{L}\Big) &= a_0\,\frac{1}{2}\cdot\sin\Big(\frac{m\pi x}{L}\Big)\\
&\quad+\sum_{n=1}^{\infty} a_n\,\cos\Big(\frac{n\pi x}{L}\Big)\cdot\sin\Big(\frac{m\pi x}{L}\Big)\\
&\quad+\sum_{n=1}^{\infty} b_n\,\sin\Big(\frac{n\pi x}{L}\Big)\cdot\sin\Big(\frac{m\pi x}{L}\Big).
\end{aligned}
$$

But $(1/2)\cdot\sin(m\pi x/L) = 0$, since the sine functions above are perpendicular to the constant functions. Also $\cos(n\pi x/L)\cdot\sin(m\pi x/L) = 0$, since all sine functions above are perpendicular to all cosine functions above. Finally $\sin(n\pi x/L)\cdot\sin(m\pi x/L) = 0$ for $m \neq n$, since sine functions with different values of m and n are mutually perpendicular. So, on the right hand side above it survives only one term, $n = m$,

$$f(x)\cdot\sin\Big(\frac{m\pi x}{L}\Big) = b_m\,\sin\Big(\frac{m\pi x}{L}\Big)\cdot\sin\Big(\frac{m\pi x}{L}\Big).$$

But on the right hand side we got the magnitude square of the sine function above,

$$\sin\Big(\frac{m\pi x}{L}\Big)\cdot\sin\Big(\frac{m\pi x}{L}\Big) = \Big\|\sin\Big(\frac{m\pi x}{L}\Big)\Big\|^2 = L.$$

Therefore,

$$f(x)\cdot\sin\Big(\frac{m\pi x}{L}\Big) = b_m\,L \quad\Rightarrow\quad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{m\pi x}{L}\Big)\,dx.$$

To get the coefficient a_m, multiply the series expansion of f by $\cos(m\pi x/L)$ and integrate on $[-L, L]$, that is

$$f(x)\cdot\cos\Big(\frac{m\pi x}{L}\Big) = \Big(\frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos\Big(\frac{n\pi x}{L}\Big) + b_n\sin\Big(\frac{n\pi x}{L}\Big)\Big)\Big)\cdot\cos\Big(\frac{m\pi x}{L}\Big).$$

As before, the linearity of the dot product together with the orthogonality properties of the basis implies that only one term survives,

$$f(x)\cdot\cos\Big(\frac{m\pi x}{L}\Big) = a_m\,\cos\Big(\frac{m\pi x}{L}\Big)\cdot\cos\Big(\frac{m\pi x}{L}\Big).$$

Since

$$\cos\Big(\frac{m\pi x}{L}\Big)\cdot\cos\Big(\frac{m\pi x}{L}\Big) = \Big\|\cos\Big(\frac{m\pi x}{L}\Big)\Big\|^2 = L,$$

we get that

$$f(x)\cdot\cos\Big(\frac{m\pi x}{L}\Big) = a_m\,L \quad\Rightarrow\quad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{m\pi x}{L}\Big)\,dx.$$

The coefficient a_0 is obtained integrating on $[-L, L]$ the series expansion for f, and using that all sine and cosine functions above are perpendicular to the constant functions, then we get

$$\int_{-L}^{L} f(x), dx = \frac{a_0}{2}\int_{-L}^{L} dx = \frac{a_0}{2}\,2L,$$

so we get the formula

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

We also skip the part of the proof about the values of the Fourier series of discontinuous functions at the point of the discontinuity. □

We now use the formulas in the Theorem above to compute the Fourier series expansion of a continuous function.

Example 7.2.3. Find the Fourier expansion of $f(x) = \begin{cases} \dfrac{x}{3}, & \text{for} \quad x \in [0,3] \\ 0, & \text{for} \quad x \in [-3,0). \end{cases}$

Solution: The Fourier expansion of f is

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

In our case $L = 3$. We start computing b_n for $n \geqslant 1$,

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3} f(x)\sin\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{3}\int_{0}^{3} \frac{x}{3}\sin\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{9}\Big(-\frac{3x}{n\pi}\cos\Big(\frac{n\pi x}{3}\Big) + \frac{9}{n^2\pi^2}\sin\Big(\frac{n\pi x}{3}\Big)\Big)\Big|_0^3 \\ &= \frac{1}{9}\Big(-\frac{9}{n\pi}\cos(n\pi) + 0 + 0 - 0\Big), \end{aligned}$$

therefore we get

$$b_n = \frac{(-1)^{(n+1)}}{n\pi}.$$

A similar calculation gives us $a_n = 0$ for $n \geqslant 1$,

$$\begin{aligned} a_n &= \frac{1}{3}\int_{-3}^{3} f(x)\cos\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{3}\int_{0}^{3} \frac{x}{3}\cos\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{1}{9}\Big(\frac{3x}{n\pi}\sin\Big(\frac{n\pi x}{3}\Big) + \frac{9}{n^2\pi^2}\cos\Big(\frac{n\pi x}{3}\Big)\Big)\Big|_0^3 \\ &= \frac{1}{9}\Big(0 + \frac{9}{n^2\pi^2}\cos(n\pi) - 0 - \frac{9}{n^2\pi^2}\Big), \end{aligned}$$

therefore we get

$$a_n = \frac{((-1)^n - 1)}{n^2\pi^2}.$$

Finally, we compute a_0,

$$a_0 = \frac{1}{3}\int_0^3 \frac{x}{3}\,dx = \frac{1}{9}\,\frac{x^2}{2}\Big|_0^3 = \frac{1}{2}.$$

Therefore, we get

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\Big[\frac{((-1)^n - 1)}{n^2\pi^2}\cos\Big(\frac{n\pi x}{3}\Big) + \frac{(-1)^{(n+1)}}{n\pi}\sin\Big(\frac{n\pi x}{3}\Big)\Big].$$

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7.2.3. Even or Odd Functions. The Fourier series expansion of a function takes a simpler form in case the function is either even or odd. More interestingly, given a function on $[0, L]$ one can extend such function to $[-L, L]$ requiring that the extension be either even or odd.

Definition 7.2.11. *A function f on $[-L, L]$ is:*

- ***even*** *iff $f(-x) = f(x)$ for all $x \in [-L, L]$;*
- ***odd*** *iff $f(-x) = -f(x)$ for all $x \in [-L, L]$.*

Remark: Not every function is either odd or even. The function $y = e^x$ is neither even nor odd. And in the case that a function is even, such as $y = \cos(x)$, or odd, such as $y = \sin(x)$, it is very simple to break that symmetry: add a constant. The functions $y = 1 + \cos(x)$ and $y = 1 + \sin(x)$ are neither even nor odd.

Below we now show that the graph of a typical even function is symmetrical about the vertical axis, while the graph of a typical odd function is symmetrical about the origin.

Example 7.2.4. The function $y = x^2$ is even, while the function $y = x^3$ is odd.

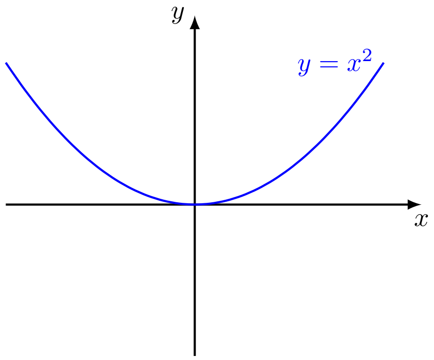

Figure 3. $y = x^2$ is even.

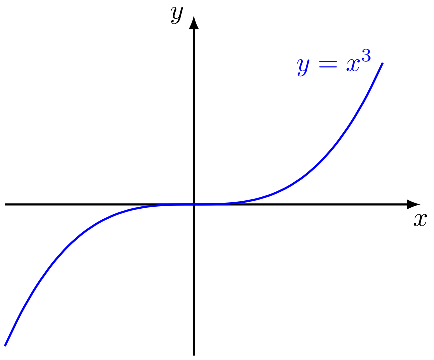

Figure 4. $y = x^3$ is odd.

◁

We now summarize a few property of even functions and odd functions.

Theorem 7.2.12. *If f_e, g_e are even and h_o, ℓ_o are odd functions, then:*

(1) $a\,f_e + b\,g_e$ is even for all $a, b \in \mathbb{R}$.
(2) $a\,h_o + b\,\ell_o$ is odd for all $a, b \in \mathbb{R}$.
(3) $f_e\,g_e$ is even.
(4) $h_o\,\ell_o$ is even.
(5) $f_e\,h_o$ is odd.
(6) $\displaystyle\int_{-L}^{L} f_e\,dx = 2\int_0^L f_e\,dx$.
(7) $\displaystyle\int_{-L}^{L} h_o\,dx = 0$.

Remark: We leave proof as an exercise. Notice that the last two equations above are simple to understand, just by looking at the figures below.

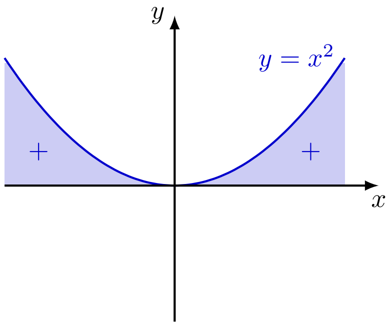

FIGURE 5. Integral of an even function.

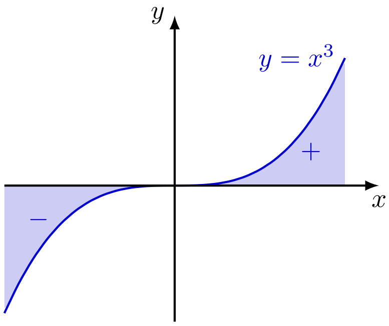

FIGURE 6. Integral of an odd function.

7.2.4. Sine and Cosine Series.

In the case that a function is either even or odd, half of its Fourier series expansion coefficients vanish. In this case the Fourier series is called either a sine or a cosine series.

Theorem 7.2.13. *Let f be a function on $[-L, L]$ with a Fourier expansion*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big].$$

(a) If f is even, then $b_n = 0$. The series Fourier series is called a cosine series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big).$$

(b) If f is odd, then $a_n = 0$. The series Fourier series is called a sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big).$$

Proof of Theorem 7.2.13:
Part (a): Suppose that f is even, then for $n \geqslant 1$ we get

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\, dx,$$

but f is even and the Sine is odd, so the integrand is odd. Therefore $b_n = 0$.
Part (b): Suppose that f is odd, then for $n \geqslant 1$ we get

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx,$$

but f is odd and the Cosine is even, so the integrand is odd. Therefore $a_n = 0$. Finally

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx,$$

but f is odd, hence $a_0 = 0$. This establishes the Theorem. □

Example 7.2.5. Find the Fourier expansion of $f(x) = \begin{cases} 1, & \text{for} \quad x \in [0, 3] \\ -1, & \text{for} \quad x \in [-3, 0). \end{cases}$

Solution: The function f is odd, so its Fourier series expansion

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

is actually a sine series. Therefore, all the coefficients $a_n = 0$ for $n \geqslant 0$. So we only need to compute the coefficients b_n. Since in our case $L = 3$, we have

$$\begin{aligned} b_n &= \frac{1}{3} \int_{-3}^{3} f(x) \sin\Big(\frac{n\pi x}{3}\Big)\, dx \\ &= \frac{1}{3} \Big(\int_{-3}^{0} (-1) \sin\Big(\frac{n\pi x}{3}\Big)\, dx + \int_{0}^{3} \sin\Big(\frac{n\pi x}{3}\Big)\, dx \Big) \\ &= \frac{2}{3} \int_{0}^{3} \sin\Big(\frac{n\pi x}{3}\Big)\, dx \\ &= \frac{2}{3} \frac{3}{n\pi} (-1) \cos\Big(\frac{n\pi x}{3}\Big)\Big|_0^3 \\ &= \frac{2}{n\pi} \big(-(-1)^n + 1 \big) \quad \Rightarrow \quad b_n = \frac{2}{n\pi} \, ((-1)^{(n+1)} + 1). \end{aligned}$$

Therefore, we get

$$f_F(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \, ((-1)^{(n+1)} + 1) \, \sin\Big(\frac{n\pi x}{L}\Big).$$

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Example 7.2.6. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} x & x \in [0, 1], \\ -x & x \in [-1, 0). \end{cases}$$

Solution: Since f is even, then $b_n = 0$. And since $L = 1$, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

We start with a_0. Since f is even, a_0 is given by

$$a_0 = 2 \int_0^1 f(x)\, dx = 2 \int_0^1 x\, dx = 2\, \frac{x^2}{2}\Big|_0^1 \quad \Rightarrow \quad a_0 = 1.$$

Now we compute the a_n for $n \geqslant 1$. Since f and the cosines are even, so is their product,

$$\begin{aligned} a_n &= 2 \int_0^1 x \cos(n\pi x)\, dx \\ &= 2 \Big(\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \Big)\Big|_0^1 \\ &= \frac{2}{n^2\pi^2} \big(\cos(n\pi) - 1 \big) \quad \Rightarrow \quad a_n = \frac{2}{n^2\pi^2} \, \big((-1)^n - 1 \big). \end{aligned}$$

So,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big((-1)^n - 1\big)\,\cos(n\pi x).$$

◁

Example 7.2.7. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 - x & x \in [0,1] \\ 1 + x & x \in [-1,0). \end{cases}$$

Solution: Since f is even, then $b_n = 0$. And since $L = 1$, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x),$$

We start computing a_0,

$$\begin{aligned} a_0 &= \int_{-1}^{1} f(x)\,dx \\ &= \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx \\ &= \Big(x + \frac{x^2}{2}\Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2}\Big)\Big|_{0}^{1} \\ &= \Big(1 - \frac{1}{2}\Big) + \Big(1 - \frac{1}{2}\Big) \quad \Rightarrow \quad a_0 = 1. \end{aligned}$$

Similarly,

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\,\cos(n\pi x)\,dx \\ &= \int_{-1}^{0} (1+x)\,\cos(n\pi x)\,dx + \int_{0}^{1} (1-x)\,\cos(n\pi x)\,dx. \end{aligned}$$

Recalling the integrals

$$\begin{aligned} \int \cos(n\pi x)\,dx &= \frac{1}{n\pi}\,\sin(n\pi x), \\ \int x\,\cos(n\pi x)\,dx &= \frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x), \end{aligned}$$

it is not difficult to see that

$$\begin{aligned} a_n &= \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{-1}^{0} + \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{-1}^{0} \\ &\quad + \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{0}^{1} - \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{0}^{1} \\ &= \Big[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\cos(-n\pi)\Big] - \Big[\frac{1}{n^2\pi^2}\cos(-n\pi) - \frac{1}{n^2\pi^2}\Big], \end{aligned}$$

we then conclude that

$$a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(-n\pi)\big] = \frac{2}{n^2\pi^2}\big(1 - (-1)^n\big).$$

So,

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big(1 - (-1)^n\big)\,\cos(n\pi x).$$

$\lhd$

7.2.5. Applications. The Fourier series expansion is a powerful tool for signal analysis. It allows us to view any signal in a different way, where several difficult problems are very simple to solve. Take sound, for example. Sounds can be transformed into electrical currents by a microphone. There are electric circuits that compute the Fourier series expansion of the currents, and the result is the frequencies and their corresponding amplitude present on that signal. Then it is possible to manipulate a precise frequency and then recombine the result into a current. That current is transformed into sound by a speaker.

This type of sound manipulation is very common. You might remember the annoying sound of the vuvuzelas—kind of loud trumpets, plastic made, very cheap—in the 2010 soccer world championship. Their sound drowned the tv commentators during the world cup. But by the 2014 world cup you could see the vuvuzelas in the stadiums but you did not hear them. It turns out vuvuzelas produce a single frequency sound, about 235 Hz. The tv equipment had incorporated a circuit that eliminated that sound, just as we described above. Fourier series expand the sound, kill that annoying frequency, and recombine the sound.

A similar, although more elaborate, sound manipulation is done constantly by sound editors in any film. Suppose you like an actor but you do not like his voice. You record the movie, then take the actor's voice, compute its Fourier series expansion, increase the amplitudes of the frequencies you like, kill the frequencies you do not like, and recombine the resulting sound. Now the actor has a new voice in the movie.

Fourier transform are used in image analysis too. A black and white image can be thought as a function from a rectangle into the set $\{0, 1\}$, that is, pixel on or pixel off. We can now write this function of two variables as a linear combination of sine and cosine functions in two space dimensions. Then one can manipulate the individual frequencies, enhancing some, decreasing others, and then recombine the result. In Fig. 7 we have an image and its Fourier transform. In Fig 8 and 9 we see the effect on the image when we erase high or low frequency modes.

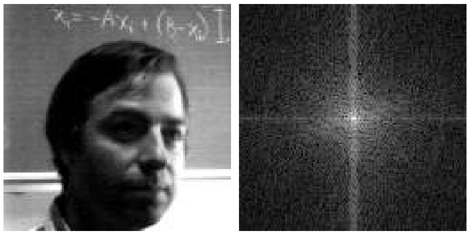

FIGURE 7. An image and its Fourier transform.

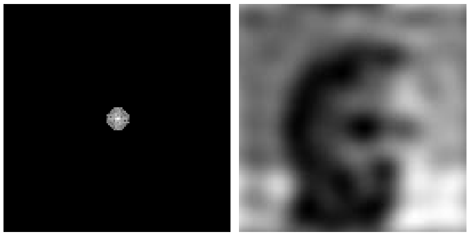

FIGURE 8. Image after erasing high frequency terms.

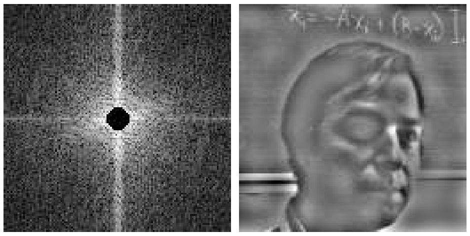

FIGURE 9. Image after erasing low frequency terms.

7.2.6. Exercises.

7.2.1.- .

7.2.2.- .

7.3. The Heat Equation

We now solve our first *partial* differential equation—the heat equation—which describes the temperature of a material as function of time and space. This is a partial differential equation because it contains partial derivatives of both time and space variables. We solve this equation using the separation of variables method, which transforms the partial differential equation into two sets of infinitely many ordinary differential equations. One set of ODEs are initial value problems, while the other set are eigenfunction problems.

The Heat equation has infinitely many solutions. One can get a unique solution imposing appropriate boundary and initial conditions. We solve the heat equation for two types of boundary conditions, called Dirichlet and Neumann conditions.

7.3.1. The Heat Equation (in One-Space Dim).

We start introducing the heat equation, for simplicity, in one-space dimension.

Definition 7.3.1. *The* ***heat equation*** *in one-space dimension, for the function u depending on t and x is*

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x), \quad \text{for} \quad t \in [0,\infty), \quad x \in [0,L],$$

where $k > 0$ is a constant and ∂_t, ∂_x are partial derivatives with respect to t and x.

Remarks:

- u is the temperature of a solid material.
- t is a time coordinate, while x is a space coordinate.
- $k > 0$ is the heat conductivity, with units $[k] = [x]^2/[t]$.
- The partial differential equation above has infinitely many solutions.
- We look for solutions satisfying both boundary conditions and initial conditions.

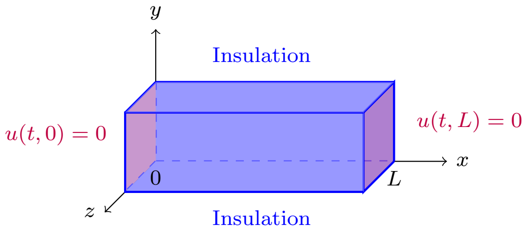

FIGURE 10. A solid bar thermally insulated on the four blue sides.

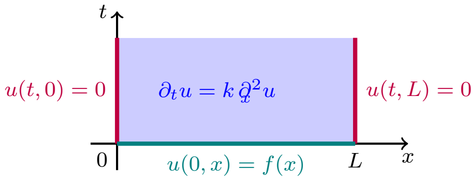

FIGURE 11. Sketch of the initial-boundary value problem on the tx-plane.

The heat equation contains partial derivatives with respect to time and space. Solving the equation means to do several integrations, which means we have a few arbitrary integration constants. So the equation has infinitely many solutions. We are going to look for solutions that satisfy some additional conditions, known as boundary conditions and initial conditions.

$$\text{Boundary Conditions:} \begin{cases} u(t,0) = 0, \\ u(t,L) = 0. \end{cases} \qquad \text{Initial Conditions:} \begin{cases} u(0,x) = f(x), \\ f(0) = f(L) = 0. \end{cases}$$

We are going to try to understand the qualitative behavior of the solutions to the heat equation before we start any detailed calculation. Recall that the heat equation is

$$\partial_t u = k\, \partial_x^2 u.$$

The meaning of the left and hand side of the equation is the following:

$$\left.\begin{array}{l}\text{How fast the temperature}\\ \text{increases of decreases.}\end{array}\right\} = k \;\; (>0) \;\; \left\{\begin{array}{l}\text{The concavity of the graph of } u\\ \text{in the variable } x \text{ at a given time.}\end{array}\right.$$

Suppose that at a fixed time $t \geqslant 0$ the graph of the temperature u as function of x is given by Fig. 12. We assume that the boundary conditions are $u(t,0) = T_0 = 0$ and $u(t,L) = T_L > 0$. Then the temperature will evolve in time following the red arrows in that figure.

The heat equation relates the time variation of the temperature, $\partial_t u$, to the curvature of the function u in the x variable, $\partial_x^2 u$. In the regions where the function u is concave up, hence $\partial_x^2 u > 0$, the heat equation says that the tempreature must increase $\partial_t u > 0$. In the regions where the function u is concave down, hence $\partial_x^2 u < 0$, the heat equation says that the tempreature must decrease $\partial_t u < 0$.

Therefore, the heat equation tries to make the temperature along the material to vary the least possible that is consistent with the boundary conditions. In the case of the figure below, the temperature will try to get to the dashed line.

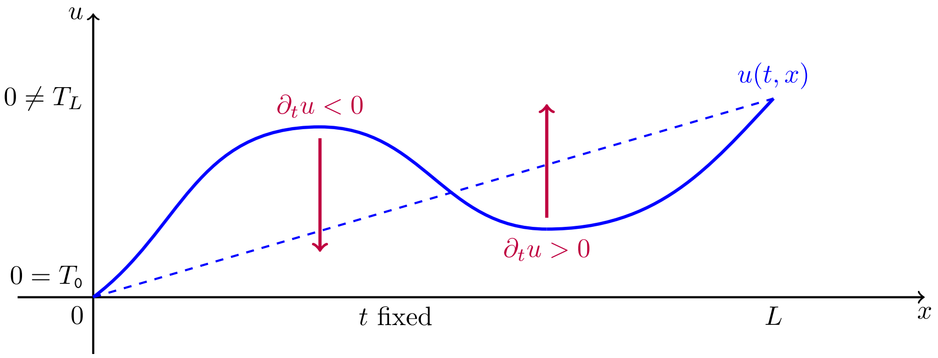

FIGURE 12. Qualitative behavior of a solution to the heat equation.

Before we start solving the heat equation we mention one generalizations and and a couple of similar equations.

- The heat equation in three space dimensions is

$$\partial_t u = k\,(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u).$$

  The method we use in this section to solve the one-space dimensional equation can be generalized to solve the three-space dimensional equation.
- The wave equation in three space dimensions is

$$\partial_t^2 u = v^2\,(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u).$$

  This equation describes how waves propagate in a medium. The constant v has units of velocity, and it is the wave speed.

- The Schrödinger equation of Quantum Mechanics is
$$i\hbar\,\partial_t u = \frac{\hbar^2}{2m}(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u) + V(t,x)\,u,$$
where m is the mass of a particle and $\hbar$ is the Planck constant divided by 2π, while $i^2 = -1$. The solutions of this equation behave more like the solutions of the wave equation than the solutions of the heat equation.

7.3.2. The IBVP: Dirichlet Conditions. We now find solutions of the one-space dimensional heat equation that satisfy a particular type of boundary conditions, called Dirichlet boundary conditions. These conditions fix the values of the temperature at two sides of the bar.

Theorem 7.3.2. *The boundary value problem for the one space dimensional heat equation,*
$$\partial_t u = k\,\partial_x^2 u, \qquad \text{BC:} \qquad u(t,0) = 0, \quad u(t,L) = 0,$$
where $k > 0$, $L > 0$ are constants, has infinitely many solutions
$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big), \quad c_n \in \mathbb{R}.$$
Furthermore, for every continuous function f on $[0,L]$ satisfying $f(0) = f(L) = 0$, there is a unique solution u of the boundary value problem above that also satisfies the initial condition
$$u(0,x) = f(x).$$
This solution u is given by the expression above, where the coefficients c_n are
$$c_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

Remarks: This is an Initial-Boundary Value Problem (IBVP). The boundary conditions are called Dirichlet boundary conditions. The physical meaning of the initial-boundary conditions is simple.

(a) The boundary conditions is to keep the temperature at the sides of the bar is constant.
(b) The initial condition is the initial temperature on the whole bar.

The proof of the IBVP above is based on the *separation of variables method*:

(1) Look for simple solutions of the boundary value problem.
(2) Any linear combination of simple solutions is also a solution. (Superposition.)
(3) Determine the free constants with the initial condition.

Proof of the Theorem 7.3.2: Look for simple solutions of the heat equation given by
$$u(t,x) = v(t)\,w(x).$$
So we look for solutions having the variables separated into two functions. Introduce this particular function in the heat equation,
$$\dot v(t)\,w(x) = k\,v(t)\,w''(x) \quad \Rightarrow \quad \frac{1}{k}\,\frac{\dot v(t)}{v(t)} = \frac{w''(x)}{w(x)},$$
where we used the notation $\dot v = dv/dt$ and $w' = dw/dx$. The separation of variables in the function u implies a separation of variables in the heat equation. The left hand side in the last equation above depends only on t and the right hand side depends only on x. The only

possible solution is that both sides are equal the same constant, call it $-\lambda$. So we end up with two equations

$$\frac{1}{k}\,\frac{\dot{v}(t)}{v(t)} = -\lambda, \qquad \text{and} \qquad \frac{w''(x)}{w(x)} = -\lambda.$$

The equation on the left is first order and simple to solve. The solution depends on λ,

$$v_\lambda(t) = c_\lambda\, e^{-k\lambda t}, \qquad c_\lambda = v_\lambda(0).$$

The second equation leads to an eigenfunction problem for w once boundary conditions are provided. These boundary conditions come from the heat equation boundary conditions,

$$\left.\begin{aligned} u(t,0) &= v(t)\,w(0) = 0 \quad \text{for all } t \geqslant 0\\ u(t,L) &= v(t)\,w(L) = 0 \quad \text{for all } t \geqslant 0 \end{aligned}\right\} \quad \Rightarrow \quad w(0) = w(L) = 0.$$

So we need to solve the following BVP for w;

$$w'' + \lambda\, w = 0, \qquad w(0) = w(L) = 0.$$

This is an eigenfunction problem we solved § 7.1, the solution is

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \quad w_n(x) = \sin\Big(\frac{n\pi x}{L}\Big), \quad n = 1, 2, \cdots.$$

Since we now know the values of λ_n, we introduce them in v_λ,

$$v_n(t) = c_n\, e^{-k(\frac{n\pi}{L})^2 t}.$$

Therefore, we got a simple solution of the heat equation BVP,

$$u_n(t,x) = c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big),$$

where $n = 1, 2, \cdots$. Since the boundary conditions for u_n are homogeneous, then any linear combination of the solutions u_n is also a solution of the heat equation with homogenous boundary conditions. Hence the function

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \sin\Big(\frac{n\pi x}{L}\Big)$$

is solution of the heat equation with homogeneous Dirichlet boundary conditions. Here the c_n are arbitrary constants. Notice that at $t = 0$ we have

$$u(0,x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big)$$

If we prescribe the c_n we get a solution u that at $t = 0$ is given by the previous formula. Is it the converse true? The answer is "yes". Given $f(x) = u(0,x)$, where $f(0) = f(L) = 0$, we can find all the coefficients c_n. Here is how: Given f on $[0,L]$, extend it to the domain $[-L,L]$ as an odd function,

$$f_{\text{odd}}(x) = f(x) \qquad \text{and} \qquad f_{\text{odd}}(-x) = -f(x), \qquad x \in [0,L]$$

Since $f(0) = 0$, we get that f_{odd} is continuous on $[-L,L]$. So f_{odd} has a Fourier series expansion. Since f_{odd} is odd, the Fourier series is a sine series

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)$$

and the coefficients are given by the formula

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{\text{odd}}(x) \sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

Since $f_{\rm odd}(x) = f(x)$ for $x \in [0, L]$, then $c_n = b_n$. This establishes the Theorem. $\square$

Example 7.3.1. Find the solution to the initial-boundary value problem

$$4\,\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 0 & x \in \big[0, \frac{2}{3}\big), \\ 5 & x \in \big[\frac{2}{3}, \frac{4}{3}\big], \\ 0 & x \in \big(\frac{4}{3}, 2\big], \end{cases} \qquad \text{BC:} \quad \begin{cases} u(t,0) = 0, \\ u(t,2) = 0. \end{cases}$$

Solution: We look for simple solutions of the form $u(t,x) = v(t)\,w(x)$,

$$4w(x)\,\frac{dv}{dt}(t) = v(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4\dot v(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda.$$

So, the equations for v and w are

$$\dot v(t) = -\frac{\lambda}{4}\,v(t), \qquad w''(x) + \lambda\,w(x) = 0.$$

The solution for v depends on λ, and is given by

$$v_\lambda(t) = c_\lambda\,e^{-\frac{\lambda}{4}t}, \qquad c_\lambda = v_\lambda(0).$$

Next we turn to the the equation for w, and we solve the BVP

$$w''(x) + \lambda\,w(x) = 0, \quad \text{with BC} \quad w(0) = w(2) = 0.$$

This is an eigenfunction problem for w and λ. This problem has solution only for $\lambda > 0$, since only in that case the characteristic polynomial has complex roots. Let $\lambda = \mu^2$, then

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\mu\,i.$$

The general solution of the differential equation is

$$w_n(x) = c_1\cos(\mu x) + c_2\sin(\mu x).$$

The first boundary conditions on w implies

$$0 = w(0) = c_1, \quad \Rightarrow \quad w(x) = c_2\,\sin(\mu x).$$

The second boundary condition on w implies

$$0 = w(2) = c_2\,\sin(\mu 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_n = \Big(\frac{n\pi}{2}\Big)^2, \qquad w_n(x) = \sin\Big(\frac{n\pi x}{2}\Big), \quad n = 1, 2, \cdots.$$

Using the values of λ_n found above in the formula for v_λ we get

$$v_n(t) = c_n\,e^{-\frac{1}{4}(\frac{n\pi}{4})^2 t}, \qquad c_n = v_n(0).$$

Therefore, we get

$$u(t,x) = \sum_{n=1}^{\infty} c_n\,e^{-(\frac{n\pi}{4})^2 t}\,\sin\Big(\frac{n\pi x}{2}\Big).$$

The initial condition is

$$f(x) = u(0,x) = \begin{cases} 0 & x \in \big[0, \frac{2}{3}\big), \\ 5 & x \in \big[\frac{2}{3}, \frac{4}{3}\big], \\ 0 & x \in \big(\frac{4}{3}, 2\big]. \end{cases}$$

We extend this function to $[-2,2]$ as an odd function, so we obtain the same sine function,

$$f_{\rm odd}(x) = f(x) \quad \text{and} \quad f_{\rm odd}(-x) = -f(x), \quad \text{where} \quad x \in [0,2].$$

The Fourier expansion of $f_{\rm odd}$ on $[-2,2]$ is a sine series

$$f_{\rm odd}(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{2}\Big).$$

The coefficients b_n are given by

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin\Big(\frac{n\pi x}{2}\Big)\,dx = \int_{2/3}^{4/3} 5 \sin\Big(\frac{n\pi x}{2}\Big)\,dx = -\frac{10}{n\pi} \cos\Big(\frac{n\pi x}{2}\Big)\Big|_{2/3}^{4/3}.$$

So we get

$$b_n = -\frac{10}{n\pi}\Big(\cos\Big(\frac{2n\pi}{3}\Big) - \cos\Big(\frac{n\pi}{3}\Big)\Big).$$

Since $f_{\rm odd}(x) = f(x)$ for $x \in [0,2]$ we get that $c_n = b_n$. So, the solution of the initial-boundary value problem for the heat equation contains is

$$u(t,x) = \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\Big(\cos\Big(\frac{n\pi}{3}\Big) - \cos\Big(\frac{2n\pi}{3}\Big)\Big)\, e^{-(\frac{n\pi}{4})^2 t} \sin\Big(\frac{n\pi x}{2}\Big).$$

$\triangleleft$

7.3.3. The IBVP: Neumann Conditions. We now find solutions of the one-space dimensional heat equation that satisfy a particular type of boundary conditions, called Neumann boundary conditions. These conditions fix the values of the heat flux at two sides of the bar.

Theorem 7.3.3. *The boundary value problem for the one space dimensional heat equation,*

$$\partial_t u = k\,\partial_x^2 u, \qquad BC: \qquad \partial_x u(t,0) = 0, \quad \partial_x u(t,L) = 0,$$

where $k > 0$, $L > 0$ are constants, has infinitely many solutions

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \cos\Big(\frac{n\pi x}{L}\Big), \quad c_n \in \mathbb{R}.$$

Furthermore, for every continuous function f on $[0,L]$ satisfying $f'(0) = f'(L) = 0$, there is a unique solution u of the boundary value problem above that also satisfies the initial condition

$$u(0,x) = f(x).$$

This solution u is given by the expression above, where the coefficients c_n are

$$c_n = \frac{2}{L} \int_0^L f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,dx, \quad n = 0,1,2,\cdots.$$

Remarks: This is an Initial-Boundary Value Problem (IBVP). The boundary conditions are called Neumann boundary conditions. The physical meaning of the initial-boundary conditions is simple.

(a) The boundary conditions keep the heat flux (proportional to $\partial_x u$) at the sides of the bar is constant.
(b) The initial condition is the initial temperature on the whole bar.

One can use Dirichlet conditions on one side and Neumann on the other side. This is called a mixed boundary condition. The proof, in all cases, is based on the separation of variables method.

Proof of the Theorem 7.3.3: Look for simple solutions of the heat equation given by

$$u(t,x) = v(t)\,w(x).$$

So we look for solutions having the variables separated into two functions. Introduce this particular function in the heat equation,

$$\dot v(t)\,w(x) = k\,v(t)\,w''(x) \quad \Rightarrow \quad \frac{1}{k}\,\frac{\dot v(t)}{v(t)} = \frac{w''(x)}{w(x)},$$

where we used the notation $\dot v = dv/dt$ and $w' = dw/dx$. The separation of variables in the function u implies a separation of variables in the heat equation. The left hand side in the last equation above depends only on t and the right hand side depends only on x. The only possible solution is that both sides are equal the same constant, call it $-\lambda$. So we end up with two equations

$$\frac{1}{k}\,\frac{\dot v(t)}{v(t)} = -\lambda, \qquad \text{and} \qquad \frac{w''(x)}{w(x)} = -\lambda.$$

The equation on the left is first order and simple to solve. The solution depends on λ,

$$v_\lambda(t) = c_\lambda\, e^{-k\lambda t}, \qquad c_\lambda = v_\lambda(0).$$

The second equation leads to an eigenfunction problem for w once boundary conditions are provided. These boundary conditions come from the heat equation boundary conditions,

$$\left.\begin{aligned} \partial_x u(t,0) = v(t)\,w'(0) = 0 \quad \text{for all } t \geqslant 0\\ \partial_x u(t,L) = v(t)\,w'(L) = 0 \quad \text{for all } t \geqslant 0 \end{aligned}\right\} \quad \Rightarrow \quad w'(0) = w'(L) = 0.$$

So we need to solve the following BVP for w;

$$w'' + \lambda\, w = 0, \qquad w'(0) = w'(L) = 0.$$

This is an eigenfunction problem, which has solutions only for $\lambda > 0$, because in that case the asociated characteristic polynomial has complex roots. If we write $\lambda = \mu^2$, for $\mu > 0$, we get the general solution

$$w(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary conditions apply on the derivative,

$$w'(x) = -\mu c_1 \sin(\mu x) + \mu c_2 \cos(\mu x).$$

The boundary conditions are

$$0 = w'(0) = \mu c_2 \quad \Rightarrow \quad c_2 = 0.$$

So the function is $w(x) = \mu c_1 \cos(\mu x)$. The second boundary condition is

$$0 = w'(L) = -\mu c_1 \sin(\mu L) \quad \Rightarrow \quad \sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi, \qquad n = 1, 2, \cdots.$$

So we get the eigenvalues and eigenfunctions

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \quad w_n(x) = \cos\Big(\frac{n\pi x}{L}\Big), \quad n = 1, 2, \cdots .$$

Since we now know the values of λ_n, we introduce them in v_λ,

$$v_n(t) = c_n\, e^{-k(\frac{n\pi}{L})^2 t}.$$

Therefore, we got a simple solution of the heat equation BVP,

$$u_n(t,x) = c_n\, e^{-k(\frac{n\pi}{L})^2 t} \cos\Big(\frac{n\pi x}{L}\Big),$$

where $n = 1, 2, \cdots$. Since the boundary conditions for u_n are homogeneous, then any linear combination of the solutions u_n is also a solution of the heat equation with homogenous boundary conditions. Hence the function

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{L})^2 t} \cos\Big(\frac{n\pi x}{L}\Big)$$

is solution of the heat equation with homogeneous Neumann boundary conditions. Notice that the constant solution $c_0/2$ is a trivial solution of the Neumann boundary value problem, which was not present in the Dirichlet boundary value problem. Here the c_n are arbitrary constants. Notice that at $t = 0$ we have

$$u(0,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\Big(\frac{n\pi x}{L}\Big)$$

If we prescribe the c_n we get a solution u that at $t = 0$ is given by the previous formula. Is it the converse true? The answer is "yes". Given $f(x) = u(0,x)$, where $f'(0) = f'(L) = 0$, we can find all the coefficients c_n. Here is how: Given f on $[0, L]$, extend it to the domain $[-L, L]$ as an even function,

$$f_{\text{even}}(x) = f(x) \qquad \text{and} \qquad f_{\text{even}}(-x) = f(x), \qquad x \in [0, L]$$

We get that f_{even} is continuous on $[-L, L]$. So f_{even} has a Fourier series expansion. Since f_{even} is even, the Fourier series is a cosine series

$$f_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big)$$

and the coefficients are given by the formula

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{\text{even}}(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx = \frac{2}{L}\int_{0}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\, dx, \qquad n = 0, 1, 2, \cdots .$$

Since $f_{\text{even}}(x) = f(x)$ for $x \in [0, L]$, then $c_n = a_n$. This establishes the Theorem. □

Example 7.3.2. Find the solution to the initial-boundary value problem

$$\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 3],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 7 & x \in [\frac{3}{2}, 3], \\ 0 & x \in [0, \frac{3}{2}), \end{cases} \qquad \text{BC:} \quad \begin{cases} u'(t,0) = 0, \\ u'(t,3) = 0. \end{cases}$$

Solution: We look for simple solutions of the form $u(t,x) = v(t)\,w(x)$,

$$w(x)\,\frac{dv}{dt}(t) = v(t)\,\frac{d^2w}{dx^2}(x) \quad\Rightarrow\quad \frac{\dot v(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda.$$

So, the equations for v and w are

$$\dot v(t) = -\lambda\, v(t), \qquad w''(x) + \lambda\, w(x) = 0.$$

The solution for v depends on λ, and is given by

$$v_\lambda(t) = c_\lambda\, e^{-\lambda t}, \qquad c_\lambda = v_\lambda(0).$$

Next we turn to the the equation for w, and we solve the BVP

$$w''(x) + \lambda\, w(x) = 0, \quad \text{with BC} \quad w'(0) = w'(3) = 0.$$

This is an eigenfunction problem for w and λ. This problem has solution only for $\lambda > 0$, since only in that case the characteristic polynomial has complex roots. Let $\lambda = \mu^2$, then

$$p(r) = r^2 + \mu^2 = 0 \quad\Rightarrow\quad r_\pm = \pm\mu\, i.$$

The general solution of the differential equation is

$$w_n(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Its derivative is

$$w'(x) = -\mu\, c_1 \sin(\mu x) + \mu\, c_2 \cos(\mu x).$$

The first boundary conditions on w implies

$$0 = w'(0) = \mu\, c_2, \quad\Rightarrow\quad c_2 = 0 \quad\Rightarrow\quad w(x) = c_1\, \cos(\mu x).$$

The second boundary condition on w implies

$$0 = w'(3) = -\mu\, c_1\, \sin(\mu 3), \quad c_1 \neq 0, \quad\Rightarrow\quad \sin(\mu 3) = 0.$$

Then, $\mu_n 3 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{3}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_n = \Big(\frac{n\pi}{3}\Big)^2, \qquad w_n(x) = \cos\Big(\frac{n\pi x}{3}\Big), \quad n = 1, 2, \cdots.$$

Using the values of λ_n found above in the formula for v_λ we get

$$v_n(t) = c_n\, e^{-(\frac{n\pi}{3})^2 t}, \qquad c_n = v_n(0).$$

Therefore, we get

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{3})^2 t}\, \cos\Big(\frac{n\pi x}{2}\Big),$$

where we have added the trivial constant solution written as $c_0/2$. The initial condition is

$$f(x) = u(0,x) = \begin{cases} 7 & x \in \big[\frac{3}{2}, 3\big], \\ 0 & x \in \big[0, \frac{3}{2}\big), \end{cases}$$

We extend f to $[-3,3]$ as an even function

$$f_{\text{even}}(x) = \begin{cases} 7 & x \in \big[\frac{3}{2}, 3\big], \\ 0 & x \in \big[-\frac{3}{2}, \frac{3}{2}\big), \\ 7 & x \in \big[-3, -\frac{3}{2}\big]. \end{cases}$$

Since f_{rmeven} is even, its Fourier expansion is a cosine series

$$f_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{3}\Big).$$

The coefficient a_0 is given by

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_{3/2}^3 7\,dx = \frac{2}{3}\,7\,\frac{3}{2} \quad\Rightarrow\quad a_0 = 7.$$

Now the coefficients a_n for $n \geqslant 1$ are given by

$$\begin{aligned} a_n &= \frac{2}{3}\int_0^3 f(x)\cos\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{2}{3}\int_{3/2}^3 7\cos\Big(\frac{n\pi x}{3}\Big)\,dx \\ &= \frac{2}{3}\,7\,\frac{3}{n\pi}\,\sin\Big(\frac{n\pi x}{3}\Big)\Big|_{3/2}^{3} \\ &= \frac{2}{3}\,7\,\frac{3}{n\pi}\,\Big(0 - \sin\Big(\frac{n\pi}{2}\Big)\Big) \\ &= -7\,\frac{2}{n\pi}\,\sin(n\pi). \end{aligned}$$

But for $n = 2k$ we have that $\sin(2k\pi/2) = \sin(k\pi) = 0$, while for $n = 2k-1$ we have that $\sin((2k-1)\pi/2) = (-1)^{k-1}$. Therefore

$$a_{2k} = 0, \qquad a_{2k-1} = 7\,\frac{2(-1)^k}{(2k-1)\pi}, k = 1, 2, \cdots.$$

We then obtain the Fourier series expansion of f_{even},

$$f_{\text{even}}(x) = \frac{7}{2} + \sum_{k=1}^{\infty} 7\frac{2(-1)^k}{(2k-1)\pi}\,\cos\Big(\frac{(2k-1)\pi x}{3}\Big)$$

But the function f has exactly the same Fourier expansion on $[0, 3]$, which means that

$$c_0 = 7, \qquad c_{2k} = 0, \qquad c_{(2k-1)} = 7\frac{2(-1)^k}{(2k-1)\pi}.$$

So the solution of the initial-boundary value problem for the heat equation is

$$u(t,x) = \frac{7}{2} + 7\sum_{k=1}^{\infty} \frac{2(-1)^k}{(2k-1)\pi}\,e^{-(\frac{(2k-1)\pi}{3})^2 t}\,\cos\Big(\frac{(2k-1)\pi x}{3}\Big).$$

◁

Example 7.3.3. Find the solution to the initial-boundary value problem

$$\partial_t u = 4\,\partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

with initial and boundary conditions given by

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: We look for simple solutions

$$u(t,x) = v(t)\,w(x).$$

We put this function into the heat equation and we get

$$w(x)\,\dot v(t) = 4\,v(t)\,w''(x) \quad \Rightarrow \quad \frac{\dot v(t)}{4\,v(t)} = \frac{w''(x)}{w(x)} = -\lambda_n.$$

The equations for v and w are

$$\dot v(t) = -4\lambda\, v(t), \qquad w''(x) + \lambda\, w(x) = 0.$$

We solve for v, which depends on the constant λ, and we get

$$v_\lambda(t) = c_\lambda\, e^{-4\lambda t},$$

where $c_\lambda = v_\lambda(0)$. Next we turn to the boundary value problem for w. We need to find the solution of

$$w''(x) + \lambda\, w(x) = 0, \quad \text{with} \quad w(0) = w(2) = 0.$$

This is an eigenfunction problem for w and λ. From § 7.1 we know that this problem has solutions only for $\lambda > 0$, which is when the characteristic polynomial of the equation for w has complex roots. So we write $\lambda = \mu^2$ for $\mu > 0$. The characteristic polynomial of the differential equation for w is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_\pm = \pm \mu i.$$

The general solution of the differential equation is

$$w(x) = \tilde c_1\, \cos(\mu x) + \tilde c_2\, \sin(\mu x).$$

The first boundary conditions on w implies

$$0 = w(0) = \tilde c_1, \quad \Rightarrow \quad w(x) = \tilde c_2\, \sin(\mu x).$$

The second boundary condition on w implies

$$0 = w(2) = \tilde c_2\, \sin(\mu 2), \quad \tilde c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = n\pi/2$, for $n \geqslant 1$. Choosing $\tilde c_2 = 1$, we conclude,

$$\lambda_m = \Big(\frac{n\pi}{2}\Big)^2, \qquad w_n(x) = \sin\Big(\frac{n\pi x}{2}\Big), \quad n = 1, 2, \cdots.$$

Using these λ_n in the expression for v_λ we get

$$v_n(t) = c_n\, e^{-4(n\pi)^2 t}$$

The expressions for v_n and w_n imply that the simple solution solution u_n has the form

$$u_n(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-4(n\pi)^2 t}\, \sin\Big(\frac{n\pi x}{2}\Big).$$

Since any linear combination of the function above is also a solution, we get

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-4(n\pi)^2 t}\, \sin\Big(\frac{n\pi x}{2}\Big).$$

The initial condition is

$$3\, \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} c_n\, \sin\Big(\frac{n\pi x}{2}\Big).$$

We now consider this function on the interval $[-2, 2]$, where is an odd function. Then, the orthogonality of these sine functions above implies

$$3 \int_{-2}^{2} \sin\Big(\frac{\pi x}{2}\Big)\, \sin\Big(\frac{m\pi x}{2}\Big)\, dx = \sum_{n=1}^{\infty} c_n \int_{-2}^{2} \sin\Big(\frac{n\pi x}{2}\Big)\, \sin\Big(\frac{m\pi x}{2}\Big)\, dx.$$

We now study the different cases:

$$c_m = 0 \quad \text{for} \quad m \neq 1.$$

Therefore we get,

$$3 \sin\Big(\frac{\pi x}{2}\Big) = c_1 \sin\Big(\frac{\pi x}{2}\Big) \quad \Rightarrow \quad c_1 = 3.$$

So the solution of the initial-boundary value problem for the heat equation is

$$u(t,x) = 3\, e^{-4\pi^2 t}\, \sin\Big(\frac{\pi x}{2}\Big).$$

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7.3.4. Exercises.

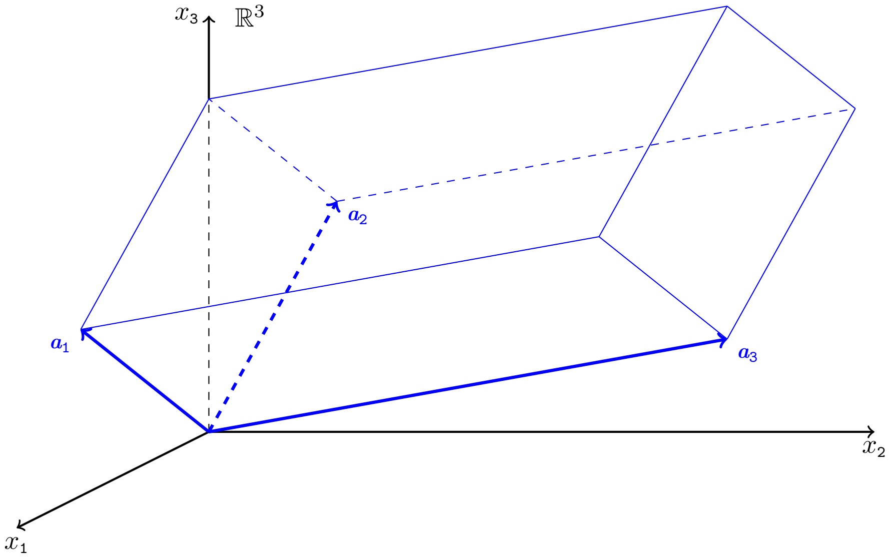
7.3.1.- .

7.3.2.- .

CHAPTER 8

Review of Linear Algebra

We review a few concepts of linear algebra, such as the Gauss operations to solve linear systems of algebraic equations, matrix operations, determinants, inverse matrix formulas, eigenvalues and eigenvectors of a matrix, diagonalizable matrices, and the exponential of a matrix.



8.1. Linear Algebraic Systems

This Section is a review of concepts from Linear Algebra needed to solve systems of linear differential equations. We start introducing an algebraic linear system, we then introduce matrices, column vectors, matrix-vector products, Gauss elimination operations, matrix echelon forms, and linear independence of vector sets.

8.1.1. Systems of Linear Equations.

The collection of results we call Linear Algebra originated with the study of linear systems of algebraic equations.

Definition 8.1.1. *An* $n \times n$ ***system of linear algebraic equations*** *is the following: Given constants* a_{ij} *and* b_i*, where* $i, j = 1 \cdots, n \geqslant 1$*, find the constants* x_j *solutions of*

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \tag{8.1.1}$$

$$\vdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n. \tag{8.1.2}$$

The system is called ***homogeneous*** *iff all sources vanish, that is,* $b_1 = \cdots = b_n = 0$.

Example 8.1.1.

(a) A 2×2 linear system on the unknowns x_1 and x_2 is the following:

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

(b) A 3×3 linear system on the unknowns x_1, x_2 and x_3 is the following:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}$$

◁

One way to find a solution to an $n \times n$ linear system is by substitution. Compute x_1 from the first equation and introduce it into all the other equations. Then compute x_2 from this new second equation and introduce it into all the remaining equations. Repeat this procedure till the last equation, where one finally obtains x_n. Then substitute back and find all the x_i, for $i = 1, \cdots, n-1$. A computational more efficient way to find a solution is to perform Gauss elimination operations on the augmented matrix of the system. Since matrix notation will simplify calculations, it is convenient we spend some time on this. We start with the basic definitions.

Definition 8.1.2. *An* $m \times n$ ***matrix***, A, *is an array of numbers*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{array}{c} m \text{ rows,} \\ n \text{ columns,'} \end{array}$$

where $a_{ij} \in \mathbb{C}$*, for* $i = 1, \cdots, m$*,* $j = 1, \cdots, n$*, are the matrix* ***coefficients****. A* ***square matrix*** *is an* $n \times n$ *matrix, and the* ***diagonal*** *coefficients in a square matrix are* a_{ii}*.*

Example 8.1.2.

(a) Examples of 2×2, 2×3, 3×2 real-valued matrices, and a 2×2 complex-valued matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \qquad D = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}.$$

(b) The coefficients of the algebraic linear systems in Example 8.1.1 can be grouped in matrices, as follows,

$$\left.\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3, \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \quad \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix}.$$

$\lhd$

Remark: A square matrix is *upper* (*lower*) triangular iff all the matrix coefficients below (above) the diagonal vanish. For example, the 3×3 matrix A below is upper triangular while B is lower triangular.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

The particular case of an $m \times 1$ matrix is called an m-vector.

Definition 8.1.3. *An **m-vector**, $\boldsymbol{v}$, is the array of numbers* $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, *where the vector **components** are $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.*

Example 8.1.3. The unknowns of the algebraic linear systems in Example 8.1.1 can be grouped in vectors, as follows,

$$\left.\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3, \end{aligned}\right\} \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}\right\} \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$\lhd$

Definition 8.1.4. *The **matrix-vector product** of an $n \times n$ matrix A and an n-vector $\boldsymbol{x}$ is an n-vector given by*

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{1n}x_n \end{bmatrix}$$

The matrix-vector product of an $n \times n$ matrix with an n-vector is another n-vector. This product is useful to express linear systems of algebraic equations in terms of matrices and vectors.

Example 8.1.4. Find the matrix-vector products for the matrices A and vectors $\boldsymbol{x}$ in Examples 8.1.2(b) and Example 8.1.3, respectively.

Solution: In the 2×2 case we get

$$A\boldsymbol{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix}.$$

In the 3×3 case we get,

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -3x_1 + x_2 + 3x_3 \\ x_2 - 4x_3 \end{bmatrix}.$$

◁

Example 8.1.5. Use the matrix-vector product to express the algebraic linear system below,

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

Solution: Introduce the coefficient matrix A, the unknown vector $\boldsymbol{x}$, and the source vector $\boldsymbol{b}$ as follows,

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Since the matrix-vector product $A\boldsymbol{x}$ is given by

$$A\boldsymbol{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix},$$

then we conclude that

$$\left.\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3, \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Leftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b}.$$

◁

It is simple to see that the result found in the Example above can be generalized to every $n \times n$ algebraic linear system.

Theorem 8.1.5 (Matrix Notation)**.** *The system in Eqs. (8.1.1)-(8.1.2) can be written as*

$$A\boldsymbol{x} = \boldsymbol{b},$$

where the coefficient matrix A, the unknown vector $\boldsymbol{x}$, and the source vector $\boldsymbol{b}$ are

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Proof of Theorem 8.1.5: From the definition of the matrix-vector product we have that

$$A\mathbf{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{1n}x_n \end{bmatrix}.$$

Then, we conclude that

$$\left.\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n, \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{1n}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \Leftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b}.$$

$\square$

We introduce one last definition, which will be helpful in the next subsection.

Definition 8.1.6. *The* ***augmented matrix*** *of* $A\boldsymbol{x} = \boldsymbol{b}$ *is the* $n \times (n+1)$ *matrix* $[A|\boldsymbol{b}]$.

The augmented matrix of an algebraic linear system contains the equation coefficients and the sources. Therefore, the augmented matrix of a linear system contains the complete information about the system.

Example 8.1.6. Find the augmented matrix of both the linear systems in Example 8.1.1.

Solution: The coefficient matrix and source vector of the first system imply that

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Rightarrow \quad [A|\boldsymbol{b}] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right].$$

The coefficient matrix and source vector of the second system imply that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 24 \\ -1 \end{bmatrix} \quad \Rightarrow \quad [A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 3 & 24 \\ 0 & 1 & -4 & -1 \end{array}\right].$$

$\lhd$

Recall that the linear combination of two vectors is defined component-wise, that is, given any numbers $a, b \in \mathbb{R}$ and any vectors $\boldsymbol{x}$, $\boldsymbol{y}$, their *linear combination* is the vector given by

$$a\boldsymbol{x} + b\boldsymbol{y} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}, \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

With this definition of linear combination of vectors it is simple to see that the matrix-vector product is a linear operation.

Theorem 8.1.7 (Linearity)**.** *The matrix-vector product is a linear operation, that is, given an* $n \times n$ *matrix* A*, then for all* n*-vectors* $\boldsymbol{x}$, $\boldsymbol{y}$ *and all numbers* $a, b \in \mathbb{R}$ *holds*

$$A(a\boldsymbol{x} + b\boldsymbol{y}) = a\,A\boldsymbol{x} + b\,A\boldsymbol{y}. \tag{8.1.3}$$

Proof of Theorem 8.1.7: Just write down the matrix-vector product in components,

$$A(a\mathbf{x} + b\mathbf{y}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix} = \begin{bmatrix} a_{11}(ax_1 + by_1) + \cdots + a_{1n}(ax_n + by_n) \\ \vdots \\ a_{n1}(ax_1 + by_1) + \cdots + a_{nn}(ax_n + by_n) \end{bmatrix}.$$

Expand the linear combinations on each component on the far right-hand side above and re-order terms as follows,

$$A(a\mathbf{x} + b\mathbf{y}) = \begin{bmatrix} a\,(a_{11}x_1 + \cdots + a_{1n}x_n) + b\,(a_{11}y_1 + \cdots + a_{1n}y_n) \\ \vdots \\ a\,(a_{n1}x_1 + \cdots + a_{nn}x_n) + b\,(a_{n1}y_1 + \cdots + a_{nn}y_n) \end{bmatrix}.$$

Separate the right-hand side above,

$$A(a\mathbf{x} + b\mathbf{y}) = a \begin{bmatrix} (a_{11}x_1 + \cdots + a_{1n}x_n) \\ \vdots \\ (a_{n1}x_1 + \cdots + a_{nn}x_n) \end{bmatrix} + b \begin{bmatrix} (a_{11}y_1 + \cdots + a_{1n}y_n) \\ \vdots \\ (a_{n1}y_1 + \cdots + a_{nn}y_n) \end{bmatrix}.$$

We then conclude that

$$A(a\mathbf{x} + b\mathbf{y}) = a\,A\boldsymbol{x} + b\,A\boldsymbol{y}.$$

This establishes the Theorem. $\square$

8.1.2. Gauss Elimination Operations. We review three operations on an augmented matrix of a linear system. *These operations change the augmented matrix of the system but they do not change the solutions of the system.* The Gauss elimination operations were already known in China around 200 BC. We call them after Carl Friedrich Gauss, since he made them very popular around 1810, when he used them to study the orbit of the asteroid Pallas, giving a systematic method to solve a 6×6 algebraic linear system.

Definition 8.1.8. *The **Gauss elimination operations** are three operations on a matrix:*

- *(i) Adding to one row a multiple of the another;*
- *(ii) Interchanging two rows;*
- *(iii) Multiplying a row by a non-zero number.*

These operations are respectively represented by the symbols given in Fig. 1.

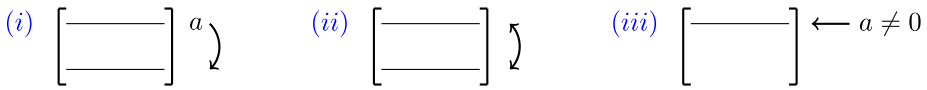


FIGURE 1. A sketch of the Gauss elimination operations.

As we said above, the Gauss elimination operations change the coefficients of the augmented matrix of a system but do not change its solution. Two systems of linear equations having the same solutions are called *equivalent*. It can be shown that there is an algorithm using these operations that transforms any $n \times n$ linear system into an equivalent system where the solutions are explicitly given.

Example 8.1.7. Find the solution to the 2×2 linear system given in Example 8.1.1 using the Gauss elimination operations.

Solution: Consider the augmented matrix of the 2×2 linear system in Example (8.1.1), and perform the following Gauss elimination operations,

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \left\{\begin{array}{c} x_1 + 0 = 1 \\ 0 + x_2 = 2 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} x_1 = 1 \\ x_2 = 2 \end{array}\right.$$

$\lhd$

Example 8.1.8. Find the solution to the 3×3 linear system given in Example 8.1.1 using the Gauss elimination operations

Solution: Consider the augmented matrix of the 3×3 linear system in Example 8.1.1 and perform the following Gauss elimination operations,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 3 & 24 \\ 0 & 1 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 7 & 6 & 27 \\ 0 & 1 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & 7 & 6 & 27 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 34 & 34 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad \begin{cases} x_1 = -6, \\ x_2 = 3, \\ x_3 = 1. \end{cases}$$

◁

In the last augmented matrix on both Examples 8.1.7 and 8.1.8 the solution is given explicitly. This is not always the case with every augmented matrix. A precise way to define the final augmented matrix in the Gauss elimination method is captured in the notion of echelon form and reduced echelon form of a matrix.

Definition 8.1.9. *An $m \times n$ matrix is in **echelon form** iff the following conditions hold:*

(i) The zero rows are located at the bottom rows of the matrix;
(ii) The first non-zero coefficient on a row is always to the right of the first non-zero coefficient of the row above it.

*The **pivot** coefficient is the first non-zero coefficient on every non-zero row in a matrix in echelon form.*

Example 8.1.9. The 6×8, 3×5 and 3×3 matrices given below are in echelon form, where the $*$ means any non-zero number and pivots are highlighted.

$$\begin{bmatrix} \boldsymbol{*} & * & * & * & * & * & * & * \\ 0 & 0 & \boldsymbol{*} & * & * & * & * & * \\ 0 & 0 & 0 & \boldsymbol{*} & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{*} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{*} & * & * & * & * \\ 0 & 0 & \boldsymbol{*} & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{*} & * & * \\ 0 & \boldsymbol{*} & * \\ 0 & 0 & \boldsymbol{*} \end{bmatrix}.$$

◁

Example 8.1.10. The following matrices are in echelon form, with pivot highlighted.

$$\begin{bmatrix} \mathbf{1} & 3 \\ 0 & \mathbf{1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 3 & 2 \\ 0 & \mathbf{4} & -2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 1 & 1 \\ 0 & \mathbf{3} & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

◁

Definition 8.1.10. *An $m \times n$ matrix is in **reduced echelon form** iff the matrix is in echelon form and the following two conditions hold:*

(i) The pivot coefficient is equal to 1;
(ii) The pivot coefficient is the only non-zero coefficient in that column.

We denote by E_A a reduced echelon form of a matrix A.

Example 8.1.11. The 6×8, 3×5 and 3×3 matrices given below are in echelon form, where the $*$ means any non-zero number and pivots are highlighted.

$$\begin{bmatrix} \mathbf{1} & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & \mathbf{1} & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \mathbf{1} & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & * & 0 & * & * \\ 0 & 0 & \mathbf{1} & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}.$$

◁

Example 8.1.12. And the following matrices are not only in echelon form but also in reduced echelon form; again, pivot coefficients are highlighted.

$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & 0 & 4 \\ 0 & \mathbf{1} & 5 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

◁

Summarizing, the Gauss elimination operations can transform any matrix into reduced echelon form. Once the augmented matrix of a linear system is written in reduced echelon form, it is not difficult to decide whether the system has solutions or not.

Example 8.1.13. Use Gauss operations to find the solution of the linear system

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -\frac{1}{2}x_1 + \frac{1}{4}x_2 &= -\frac{1}{4}. \end{aligned}$$

Solution: We find the system augmented matrix and perform appropriate Gauss elimination operations,

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

From the last augmented matrix above we see that the original linear system has the same solutions as the linear system given by

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ 0 &= 1. \end{aligned}$$

Since the latter system has no solutions, the original system has no solutions. ◁

The situation shown in Example 8.1.13 is true in general. If the augmented matrix $[A|\boldsymbol{b}]$ of an algebraic linear system is transformed by Gauss operations into the augmented matrix $[\tilde{A}|\tilde{\boldsymbol{b}}]$ having a row of the form $[0, \cdots, 0|1]$, then the original algebraic linear system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution.

Example 8.1.14. Find all vectors $\boldsymbol{b}$ such that the system $A\boldsymbol{x} = \boldsymbol{b}$ has solutions, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

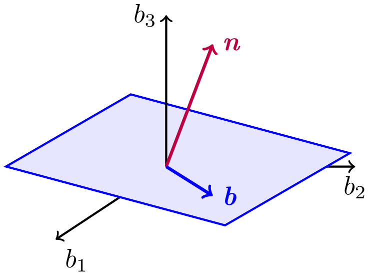
Solution: We do not need to write down the algebraic linear system, we only need its augmented matrix,

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_1 + b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 3 & -3 & b_3 - 2b_1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_3 + b_1 + 3b_2 \end{array}\right].$$

Therefore, the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has solutions $\Leftrightarrow$ the source vector satisfies the equation holds $b_1 + 3b_2 + b_3 = 0$.
That is, every source vector $\boldsymbol{b}$ that lie on the plane normal to the vector $\boldsymbol{n}$ is a source vector such that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has solution, where

$$\boldsymbol{n} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$



$\lhd$

8.1.3. Linearly Dependence. We generalize the idea of two vectors lying on the same line, and three vectors lying on the same plane, to an arbitrary number of vectors.

Definition 8.1.11. *A set of vectors* $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$, *with* $k \geqslant 1$ *is called* ***linearly dependent*** *iff there exists constants* $c_1, \cdots, c_k$, *with at least one of them non-zero, such that*

$$c_1\, \boldsymbol{v}_1 + \cdots + c_k\, \boldsymbol{v}_k = \boldsymbol{0}. \tag{8.1.4}$$

The set of vectors is called ***linearly independent*** *iff it is not linearly dependent, that is, the only constants* $c_1, \cdots, c_k$ *that satisfy Eq. (8.1.4) are given by* $c_1 = \cdots = c_k = 0$.

In other words, a set of vectors is linearly dependent iff one of the vectors is a linear combination of the other vectors. When this is not possible, the set is called linearly independent.

Example 8.1.15. Show that the following set of vectors is linearly dependent,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\},$$

and express one of the vectors as a linear combination of the other two.

Solution: We need to find constants c_1, c_2, and c_3 solutions of the equation

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} c_1 + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} c_2 + \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution to this linear system can be obtained with Gauss elimination operations,

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 \\ 0 & -4 & 4 \\ 0 & -8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} c_1 = -2c_3, \\ c_2 = c_3, \\ c_3 = \text{ free.} \end{cases}$$

Since there are non-zero constants c_1, c_2, c_3 solutions to the linear system above, the vectors are linearly dependent. Choosing $c_3 = -1$ we obtain the third vector as a linear combination of the other two vectors,

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

$\lhd$

8.1.4. Exercises.

8.1.1.- .

8.1.2.- .

8.2. Matrix Algebra

The matrix-vector product introduced in Section 8.1 implies that an $n \times n$ matrix A is a function $A : \mathbb{R}^n \to \mathbb{R}^n$. This idea leads to introduce matrix operations, like the operations introduced for functions $f : \mathbb{R} \to \mathbb{R}$. These operations include linear combinations of matrices; the composition of matrices, also called matrix multiplication; the inverse of a square matrix; and the determinant of a square matrix. These operations will be needed in later Sections when we study systems of differential equations.

8.2.1. A Matrix is a Function. The matrix-vector product leads to the interpretation that an $n \times n$ matrix A is a function. If we denote by $\mathbb{R}^n$ the space of all n-vectors, we see that the matrix-vector product associates to the n-vector $\boldsymbol{x}$ the unique n-vector $\boldsymbol{y} = A\boldsymbol{x}$. Therefore the matrix A determines a function $A : \mathbb{R}^n \to \mathbb{R}^n$.

Example 8.2.1. Describe the action on $\mathbb{R}^2$ of the function given by the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{8.2.1}$$

Solution: The action of this matrix on an arbitrary element $\boldsymbol{x} \in \mathbb{R}^2$ is given below,

$$A\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

Therefore, this matrix interchanges the components x_1 and x_2 of the vector $\boldsymbol{x}$. It can be seen in the first picture in Fig. 2 that this action can be interpreted as a reflection on the plane along the line $x_1 = x_2$. ◁

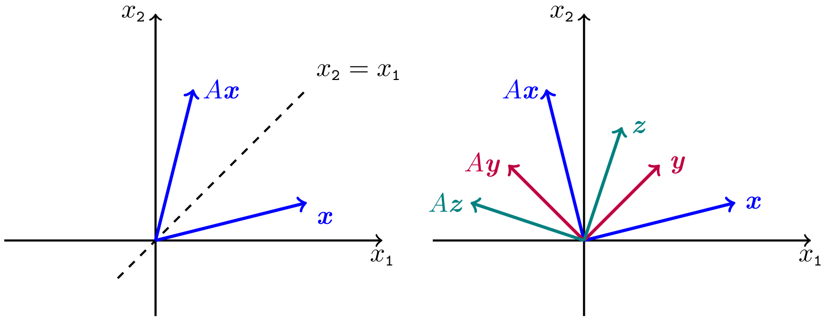


FIGURE 2. Geometrical meaning of the function determined by the matrix in Eq. (8.2.1) and the matrix in Eq. (8.2.2), respectively.

Example 8.2.2. Describe the action on $\mathbb{R}^2$ of the function given by the 2×2 matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.2.2}$$

Solution: The action of this matrix on an arbitrary element $\boldsymbol{x} \in \mathbb{R}^2$ is given below,

$$A\boldsymbol{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

In order to understand the action of this matrix, we give the following particular cases:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

These cases are plotted in the second figure on Fig. 2, and the vectors are called $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively. We therefore conclude that this matrix produces a ninety degree counterclockwise rotation of the plane. ◁

An example of a scalar-valued function is $f : \mathbb{R} \to \mathbb{R}$. We have seen here that an $n \times n$ matrix A is a function $A : \mathbb{R}^n \to \mathbb{R}^n$. Therefore, one can think an $n \times n$ matrix as a generalization of the concept of function from $\mathbb{R}$ to $\mathbb{R}^n$, for any positive integer n. It is well-known how to define several operations on scalar-valued functions, like linear combinations, compositions, and the inverse function. Therefore, it is reasonable to ask if these operation on scalar-valued functions can be generalized as well to matrices. The answer is yes, and the study of these and other operations is the subject of the rest of this Section.

8.2.2. Matrix Operations.

The linear combination of matrices refers to the addition of two matrices and the multiplication of a matrix by scalar. Linear combinations of matrices are defined component by component. For this reason we introduce the component notation for matrices and vectors. We denote an $m \times n$ matrix by $A = [A_{ij}]$, where A_{ij} are the components of matrix A, with $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Analogously, an n-vector is denoted by $\boldsymbol{v} = [v_j]$, where v_j are the components of the vector $\boldsymbol{v}$. We also introduce the notation $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$, that is, the set $\mathbb{F}$ can be the real numbers or the complex numbers.

Definition 8.2.1. *Let $A = [A_{ij}]$ and $B = [B_{ij}]$ be $m \times n$ matrices in $\mathbb{F}^{m,n}$ and a, b be numbers in $\mathbb{F}$. The **linear combination** of A and B is also and $m \times n$ matrix in $\mathbb{F}^{m,n}$, denoted as $a\,A + b\,B$, and given by*

$$a\,A + b\,B = [a\,A_{ij} + b\,B_{ij}].$$

The particular case where $a = b = 1$ corresponds to the addition of two matrices, and the particular case of $b = 0$ corresponds to the multiplication of a matrix by a number, that is,

$$A + B = [A_{ij} + B_{ij}], \qquad a\,A = [a\,A_{ij}].$$

Example 8.2.3. Find the $A + B$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$.

Solution: The addition of two equal size matrices is performed component-wise:

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$

◁

Example 8.2.4. Find the $A + B$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: The matrices have different sizes, so their addition is not defined. ◁

Example 8.2.5. Find $2A$ and $A/3$, where $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Solution: The multiplication of a matrix by a number is done component-wise, therefore

$$2A = 2\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix} = \begin{bmatrix}2 & 6 & 10\\ 4 & 8 & 12\end{bmatrix}, \qquad \frac{A}{3} = \frac{1}{3}\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\[6pt] \frac{2}{3} & \frac{4}{3} & 2\end{bmatrix}.$$

◁

Since matrices are generalizations of scalar-valued functions, one can define operations on matrices that, unlike linear combinations, have no analogs on scalar-valued functions. One of such operations is the transpose of a matrix, which is a new matrix with the rows and columns interchanged.

Definition 8.2.2. *The* ***transpose*** *of a matrix* $A = [A_{ij}] \in \mathbb{F}^{m,n}$ *is the matrix denoted as* $A^T = \big[(A^T)_{kl}\big] \in \mathbb{F}^{n,m}$, *with its components given by* $\big(A^T\big)_{kl} = A_{lk}$.

Example 8.2.6. Find the transpose of the 2×3 matrix $A = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$.

Solution: Matrix A has components A_{ij} with $i = 1, 2$ and $j = 1, 2, 3$. Therefore, its transpose has components $(A^T)_{ji} = A_{ij}$, that is, A^T has three rows and two columns,

$$A^T = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}.$$

◁

If a matrix has complex-valued coefficients, then the conjugate of a matrix can be defined as the conjugate of each component.

Definition 8.2.3. *The* ***complex conjugate*** *of a matrix* $A = [A_{ij}] \in \mathbb{F}^{m,n}$ *is the matrix* $\overline{A} = \big[\overline{A}_{ij}\big] \in \mathbb{F}^{m,n}$.

Example 8.2.7. A matrix A and its conjugate is given below,

$$A = \begin{bmatrix}1 & 2+i\\ -i & 3-4i\end{bmatrix}, \quad \Leftrightarrow \quad \overline{A} = \begin{bmatrix}1 & 2-i\\ i & 3+4i\end{bmatrix}.$$

◁

Example 8.2.8. A matrix A has real coefficients iff $A = \overline{A}$; It has purely imaginary coefficients iff $A = -\overline{A}$. Here are examples of these two situations:

$$A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \quad \Rightarrow \quad \overline{A} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = A;$$

$$A = \begin{bmatrix}i & 2i\\ 3i & 4i\end{bmatrix} \quad \Rightarrow \quad \overline{A} = \begin{bmatrix}-i & -2i\\ -3i & -4i\end{bmatrix} = -A.$$

◁

Definition 8.2.4. *The* ***adjoint*** *of a matrix* $A \in \mathbb{F}^{m,n}$ *is the matrix* $A^* = \overline{A}^{\,T} \in \mathbb{F}^{n,m}$.

Since $\big(\overline{A}\big)^T = \overline{(A^T)}$, the order of the operations does not change the result, that is why there is no parenthesis in the definition of A^*.

Example 8.2.9. A matrix A and its adjoint is given below,

$$A = \begin{bmatrix} 1 & 2+i \\ -i & 3-4i \end{bmatrix}, \quad \Leftrightarrow \quad A^* = \begin{bmatrix} 1 & i \\ 2-i & 3+4i \end{bmatrix}.$$

◁

The transpose, conjugate and adjoint operations are useful to specify certain classes of matrices with particular symmetries. Here we introduce few of these classes.

Definition 8.2.5. *An $n \times n$ matrix A is called:*
(a) ***symmetric*** *iff holds $A = A^T$;*
(b) ***skew-symmetric*** *iff holds $A = -A^T$;*
(c) ***Hermitian*** *iff holds $A = A^*$;*
(d) ***skew-Hermitian*** *iff holds $A = -A^*$.*

Example 8.2.10. We present examples of each of the classes introduced in Def. 8.2.5.

Part (a): Matrices A and B are symmetric. Notice that A is also Hermitian, while B is not Hermitian,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 3 & 4 & 8 \end{bmatrix} = A^T, \qquad B = \begin{bmatrix} 1 & 2+3i & 3 \\ 2+3i & 7 & 4i \\ 3 & 4i & 8 \end{bmatrix} = B^T.$$

Part (b): Matrix C is skew-symmetric,

$$C = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} \quad \Rightarrow \quad C^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = -C.$$

Notice that the diagonal elements in a skew-symmetric matrix must vanish, since $C_{ij} = -C_{ji}$ in the case $i = j$ means $C_{ii} = -C_{ii}$, that is, $C_{ii} = 0$.

Part (c): Matrix D is Hermitian but is not symmetric:

$$D = \begin{bmatrix} 1 & 2+i & 3 \\ 2-i & 7 & 4+i \\ 3 & 4-i & 8 \end{bmatrix} \quad \Rightarrow \quad D^T = \begin{bmatrix} 1 & 2-i & 3 \\ 2+i & 7 & 4-i \\ 3 & 4+i & 8 \end{bmatrix} \neq D,$$

however,

$$D^* = \overline{D}^T = \begin{bmatrix} 1 & 2+i & 3 \\ 2-i & 7 & 4+i \\ 3 & 4-i & 8 \end{bmatrix} = D.$$

Notice that the diagonal elements in a Hermitian matrix must be real numbers, since the condition $A_{ij} = \overline{A}_{ji}$ in the case $i = j$ implies $A_{ii} = \overline{A}_{ii}$, that is, $2i\mathrm{Im}(A_{ii}) = A_{ii} - \overline{A}_{ii} = 0$. We can also verify what we said in part (a), matrix A is Hermitian since $A^* = \overline{A}^T = A^T = A$.

Part (d): The following matrix E is skew-Hermitian:

$$E = \begin{bmatrix} i & 2+i & -3 \\ -2+i & 7i & 4+i \\ 3 & -4+i & 8i \end{bmatrix} \quad \Rightarrow \quad E^T = \begin{bmatrix} i & -2+i & 3 \\ 2+i & 7i & -4+i \\ -3 & 4+i & 8i \end{bmatrix}$$

therefore,

$$E^* = \overline{E}^T \begin{bmatrix} -i & -2-i & 3 \\ 2-i & -7i & -4-i \\ -3 & 4-i & -8i \end{bmatrix} = -E.$$

A skew-Hermitian matrix has purely imaginary elements in its diagonal, and the off diagonal elements have skew-symmetric real parts with symmetric imaginary parts. ◁

The trace of a square matrix is a number, the sum of all the diagonal elements of the matrix.

Definition 8.2.6. *The* ***trace*** *of a square matrix* $A = \big[A_{ij}\big] \in \mathbb{F}^{n,n}$*, denoted as* $\operatorname{tr}(A) \in \mathbb{F}$*, is the sum of its diagonal elements, that is, the scalar given by* $\operatorname{tr}(A) = A_{11} + \cdots + A_{nn}$*.*

Example 8.2.11. Find the trace of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: We only have to add up the diagonal elements:

$$\operatorname{tr}(A) = 1 + 5 + 9 \quad \Rightarrow \quad \operatorname{tr}(A) = 15.$$

◁

The operation of matrix multiplication originates in the composition of functions. We call it matrix multiplication instead of matrix composition because it reduces to the multiplication of real numbers in the case of 1×1 real matrices. Unlike the multiplication of real numbers, the product of general matrices is not commutative, that is, $AB \neq BA$ in the general case. This property reflects the fact that the composition of two functions is a non-commutative operation.

Definition 8.2.7. *The* ***matrix multiplication*** *of the* $m \times n$ *matrix* $A = [A_{ij}]$ *and the* $n \times \ell$ *matrix* $B = [B_{jk}]$*, where* $i = 1, \cdots, m$*,* $j = 1, \cdots, n$ *and* $k = 1, \cdots, \ell$*, is the* $m \times \ell$ *matrix* AB *given by*

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}. \tag{8.2.3}$$

The product is not defined for two arbitrary matrices, since the size of the matrices is important: The numbers of columns in the first matrix must match the numbers of rows in the second matrix.

$$\begin{array}{ccccc} A & \text{times} & B & \text{defines} & AB \\ m \times \boldsymbol{n} & & \boldsymbol{n} \times \ell & & m \times \ell \end{array}$$

Example 8.2.12. Compute AB, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: The component $(AB)_{11} = 4$ is obtained from the first row in matrix A and the first column in matrix B as follows:

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{3} & 0 \\ \mathbf{2} & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & 1 \\ 1 & -2 \end{bmatrix}, \qquad (2)(3) + (-1)(2) = 4;$$

The component $(AB)_{12} = -1$ is obtained as follows:

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & \mathbf{0} \\ 2 & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} 4 & \mathbf{1} \\ 1 & -2 \end{bmatrix}, \qquad (2)(0) + (-1)(1) = -1;$$

The component $(AB)_{21} = 1$ is obtained as follows:

$$\begin{bmatrix} 2 & -1 \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{3} & 0 \\ \mathbf{2} & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ \mathbf{1} & -2 \end{bmatrix}, \qquad (-1)(3) + (2)(2) = 1;$$

And finally the component $(AB)_{22} = -2$ is obtained as follows:

$$\begin{bmatrix} 2 & -1 \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} 3 & \mathbf{0} \\ 2 & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & \mathbf{-2} \end{bmatrix}, \qquad (-1)(0) + (2)(-1) = -2.$$

◁

Example 8.2.13. Compute BA, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution: We find that $BA = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}$. Notice that in this case $AB \neq BA$. ◁

Example 8.2.14. Compute AB and BA, where $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: The product AB is

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Rightarrow \quad AB = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

The product BA is not possible. ◁

Example 8.2.15. Compute AB and BA, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

Solution: We find that

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

$$BA = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Remarks:

(a) Notice that in this case $AB \neq BA$.
(b) Notice that $BA = 0$ but $A \neq 0$ and $B \neq 0$.

◁

8.2.3. The Inverse Matrix.

We now introduce the concept of the inverse of a square matrix. Not every square matrix is invertible. The inverse of a matrix is useful to compute solutions to linear systems of algebraic equations.

Definition 8.2.8. *The matrix $I_n \in \mathbb{F}^{n,n}$ is the* ***identity matrix*** *iff $I_n \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{F}^n$.*

It is simple to see that the components of the identity matrix are given by

$$I_n = [I_{ij}] \quad \text{with} \quad \begin{cases} I_{ii} = 1 \\ I_{ij} = 0 \quad i \neq j. \end{cases}$$

The cases $n = 2, 3$ are given by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 8.2.9. *A matrix $A \in \mathbb{F}^{n,n}$ is called* ***invertible*** *iff there exists a matrix, denoted as A^{-1}, such that $(A^{-1})A = I_n$, and $A(A^{-1}) = I_n$.*

Example 8.2.16. Verify that the matrix and its inverse are given by

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

Solution: We have to compute the products,

$$A\big(A^{-1}\big) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Rightarrow \quad A\big(A^{-1}\big) = I_2.$$

It is simple to check that the equation $\big(A^{-1}\big)A = I_2$ also holds. $\lhd$

Theorem 8.2.10. *Given a* 2×2 *matrix* A *introduce the number* Δ *as follows,*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \Delta = ad - bc.$$

The matrix A *is invertible iff* $\Delta \neq 0$. *Furthermore, if* A *is invertible, its inverse is given by*

$$\mathsf{A}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{8.2.4}$$

The number Δ is called the determinant of A, since it is the number that determines whether A is invertible or not.

Example 8.2.17. Compute the inverse of matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, given in Example 8.2.16.

Solution: Following Theorem 8.2.10 we first compute $\Delta = 6 - 4 = 4$. Since $\Delta \neq 0$, then A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

$\lhd$

Example 8.2.18. Compute the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution: Following Theorem 8.2.10 we first compute $\Delta = 6 - 6 = 0$. Since $\Delta = 0$, then matrix A is not invertible. $\lhd$

The matrix operations we have introduced are useful to solve matrix equations, where the unknown is a matrix. We now show an example of a matrix equation.

Example 8.2.19. Find a matrix X such that $AXB = I$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution: There are many ways to solve a matrix equation. We choose to multiply the equation by the inverses of matrix A and B, if they exist. So first we check whether A is invertible. But

$$\det(A) = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0,$$

so A is indeed invertible. Regarding matrix B we get

$$\det(B) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0,$$

so B is also invertible. We then compute their inverses,

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}, \qquad B = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

We can now compute X,

$$AXB = I \quad \Rightarrow \quad A^{-1}(AXB)B^{-1} = A^{-1}IB^{-1} \quad \Rightarrow \quad X = A^{-1}B^{-1}.$$

Therefore,

$$X = \frac{1}{-5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = -\frac{1}{15}\begin{bmatrix} 5 & -7 \\ -5 & 4 \end{bmatrix}$$

so we obtain

$$X = \begin{bmatrix} -\frac{1}{3} & \frac{7}{15} \\ \frac{1}{3} & -\frac{4}{15} \end{bmatrix}.$$

◁

8.2.4. Computing the Inverse Matrix.

Gauss operations can be used to compute the inverse of a matrix. The reason for this is simple to understand in the case of 2×2 matrices, as can be seen in the following Example.

Example 8.2.20. Given any 2×2 matrix A, find its inverse matrix, A^{-1}, or show that the inverse does not exist.

Solution: If the inverse matrix, A^{-1} exists, then denote it as $A^{-1} = [\boldsymbol{x}_1,\ \boldsymbol{x}_2]$. The equation $A(A^{-1}) = I_2$ is then equivalent to $A\,[\boldsymbol{x}_1,\ \boldsymbol{x}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This equation is equivalent to solving two algebraic linear systems,

$$A\,\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A\,\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Here is where we can use Gauss elimination operations. We use them on both systems

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right], \qquad \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}\right].$$

However, we can solve both systems at the same time if we do Gauss operations on the bigger augmented matrix

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right].$$

Now, perform Gauss operations until we obtain the reduced echelon form for $[A|I_2]$. Then we can have two different types of results:

- If there is no line of the form $[0, 0|*, *]$ with any of the star coefficients non-zero, then matrix A is invertible and the solution vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ form the columns of the inverse matrix, that is, $A^{-1} = [\boldsymbol{x}_1, \boldsymbol{x}_2]$.
- If there is a line of the form $[0, 0|*, *]$ with any of the star coefficients non-zero, then matrix A is not invertible. ◁

Example 8.2.21. Use Gauss operations to find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution: As we said in the Example above, perform Gauss operation on the augmented matrix $[A|I_2]$ until the reduced echelon form is obtained, that is,

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -4 & 1 & -2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

That is, matrix A is invertible and the inverse is

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad \Leftrightarrow \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

◁

Example 8.2.22. Use Gauss operations to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$.

Solution: We perform Gauss operations on the augmented matrix $[A|I_3]$ until we obtain its reduced echelon form, that is,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

We conclude that matrix A is invertible and

$$A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

◁

8.2.5. Overview of Determinants.

A determinant is a scalar computed form a square matrix that gives important information about the matrix, for example if the matrix is invertible or not. We now review the definition and properties of the determinant of 2×2 and 3×3 matrices.

Definition 8.2.11. *The **determinant of a** 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

*The **determinant of a** 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 8.2.23. The following three examples show that the determinant can be a negative, zero or positive number.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \qquad \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5, \qquad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0.$$

The following is an example shows how to compute the determinant of a 3×3 matrix,

$$\begin{aligned}\begin{vmatrix}1 & 3 & -1\\ 2 & 1 & 1\\ 3 & 2 & 1\end{vmatrix} &= (1)\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix} - 3\begin{vmatrix}2 & 1\\ 3 & 1\end{vmatrix} + (-1)\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}\\ &= (1-2) - 3\,(2-3) - (4-3)\\ &= -1 + 3 - 1\\ &= 1.\end{aligned}$$

$\triangleleft$

Remark: The determinant of upper or lower triangular matrices is the product of the diagonal coefficients.

Example 8.2.24. Compute the determinant of a 3×3 upper triangular matrix.

Solution:

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{vmatrix} = a_{11}\begin{vmatrix}a_{22} & a_{23}\\ 0 & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}0 & a_{23}\\ 0 & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}0 & a_{22}\\ 0 & 0\end{vmatrix} = a_{11}a_{22}a_{33}.$$

$\triangleleft$

The absolute value of the determinant of a 2×2 matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2]$ has a geometrical meaning: It is the area of the parallelogram whose sides are given by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, that is, by the columns of the matrix A; see Fig. 3. Analogously, the absolute value of the determinant of a 3×3 matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ also has a geometrical meaning: It is the volume of the parallelepiped whose sides are given by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, that is, by the columns of the matrix A; see Fig. 3.

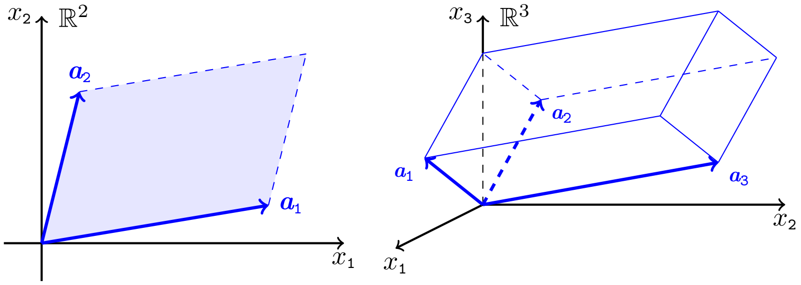


FIGURE 3. Geometrical meaning of the determinant.

The determinant of an $n \times n$ matrix A can be defined generalizing the properties that areas of parallelogram have in two dimensions and volumes of parallelepipeds have in three dimensions. One of these properties is the following: if one of the column vectors of the matrix A is a linear combination of the others, then the figure determined by these column vectors is not n-dimensional but $(n-1)$-dimensional, so its volume must vanish. We highlight this property of the determinant of $n \times n$ matrices in the following result.

Theorem 8.2.12. *The set of n-vectors $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$, with $n \geqslant 1$, is linearly dependent iff*

$$\det[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n] = 0.$$

Example 8.2.25. Show whether the set of vectors below linearly independent,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} \right\}.$$

The determinant of the matrix whose column vectors are the vectors above is given by

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & 2 & 2 \\ 3 & 1 & 7 \end{vmatrix} = (1)\,(14-2) - 3\,(14-6) + (-3)\,(2-6) = 12 - 24 + 12 = 0.$$

Therefore, the set of vectors above is linearly dependent. $\lhd$

The determinant of a square matrix also determines whether the matrix is invertible or not.

Theorem 8.2.13. *An $n \times n$ matrix A is invertible iff holds $\det(A) \neq 0$.*

Example 8.2.26. Is matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$ invertible?

Solution: We only need to compute the determinant of A.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{vmatrix} = (1) \begin{vmatrix} 5 & 7 \\ 7 & 9 \end{vmatrix} - (2) \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} + (3) \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}.$$

Using the definition of determinant for 2 × 2 matrices we obtain

$$\det(A) = (45 - 49) - 2(18 - 21) + 3(14 - 15) = -4 + 6 - 3.$$

Since $\det(A) = -1$, that is, non-zero, matrix A is invertible. $\lhd$

8.2.6. Exercises.

8.2.1.- .

8.2.2.- .

8.3. Eigenvalues and Eigenvectors

We continue with the review on Linear Algebra we started in § 8.1 and § 8.2. We saw that a square matrix is a function on the space of vectors, since it acts on a vector and the result is another vector. In this section we see that, given an $n \times n$ matrix, there may exist lines through the origin in $\mathbb{R}^n$ that are left invariant under the action of the matrix. This means that such a matrix acting on any vector in such a line is a vector on the same line. The vector is called an eigenvector of the matrix, and the proportionality factor is called an eigenvalue. If there is a linearly independent set of n eigenvectors of an $n \times n$ matrix, then this matrix is diagonalizable. In § 8.4 we will see that the exponential of a matrix is particularly simple to compute in the case that the matrix is diagonalizable. In Chapter 5 we use the exponential of a matrix to write the solutions of systems of linear differential equations with constant coefficients.

8.3.1. Eigenvalues and Eigenvectors.

When a square matrix acts on a vector the result is another vector that, more often than not, points in a different direction from the original vector. However, there may exist vectors whose direction is not changed by the matrix. These will be important for us, so we give them a name.

Definition 8.3.1. *A number λ and a nonzero n-vector $\boldsymbol{v}$ are an* ***eigenvalue*** *with corresponding* ***eigenvector*** *(eigenpair) of an $n \times n$ matrix A iff they satisfy the equation*

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

Remark: We see that an eigenvector $\boldsymbol{v}$ determines a particular direction in the space $\mathbb{R}^n$, given by $(a\boldsymbol{v})$ for $a \in \mathbb{R}$, that remains invariant under the action of the matrix A. That is, the result of matrix A acting on any vector $(a\boldsymbol{v})$ on the line determined by $\boldsymbol{v}$ is again a vector on the same line, since

$$A(a\boldsymbol{v}) = a\,A\boldsymbol{v} = a\lambda\boldsymbol{v} = \lambda(a\boldsymbol{v}).$$

Example 8.3.1. Verify that the pair λ_1, $\boldsymbol{v}_1$ and the pair λ_2, $\boldsymbol{v}_2$ are eigenvalue and eigenvector pairs of matrix A given below,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \begin{cases} \lambda_1 = 4 & \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \lambda_2 = -2 & \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{cases}$$

Solution: We just must verify the definition of eigenvalue and eigenvector given above. We start with the first pair,

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \boldsymbol{v}_1 \quad \Rightarrow \quad A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1.$$

A similar calculation for the second pair implies,

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \boldsymbol{v}_2 \quad \Rightarrow \quad A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2.$$

◁

Example 8.3.2. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: This is the matrix given in Example 8.2.1. The action of this matrix on the plane is a reflection along the line $x_1 = x_2$, as it was shown in Fig. 2. Therefore, this line $x_1 = x_2$ is left invariant under the action of this matrix. This property suggests that an eigenvector is any vector on that line, for example

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 1.$$

So, we have found one eigenvalue-eigenvector pair: $\lambda_1 = 1$, with $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We remark that any nonzero vector proportional to $\boldsymbol{v}_1$ is also an eigenvector. Another choice fo eigenvalue-eigenvector pair is $\lambda_1 = 1$, with $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. It is not so easy to find a second eigenvector which does not belong to the line determined by $\boldsymbol{v}_1$. One way to find such eigenvector is noticing that the line perpendicular to the line $x_1 = x_2$ is also left invariant by matrix A. Therefore, any nonzero vector on that line must be an eigenvector. For example the vector $\boldsymbol{v}_2$ below, since

$$\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_2 = -1.$$

So, we have found a second eigenvalue-eigenvector pair: $\lambda_2 = -1$, with $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These two eigenvectors are displayed on Fig. 4. $\triangleleft$

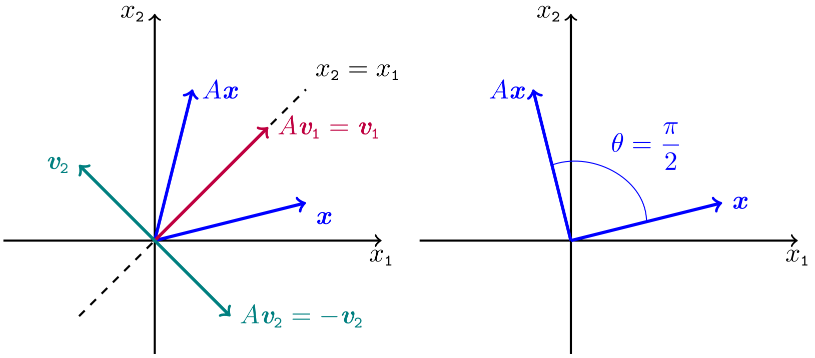


FIGURE 4. The first picture shows the eigenvalues and eigenvectors of the matrix in Example 8.3.2. The second picture shows that the matrix in Example 8.3.3 makes a counterclockwise rotation by an angle θ, which proves that this matrix does not have eigenvalues or eigenvectors.

There exist matrices that do not have eigenvalues and eigenvectors, as it is show in the example below.

Example 8.3.3. Fix any number $\theta \in (0, 2\pi)$ and define the matrix $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Show that A has no real eigenvalues.

Solution: One can compute the action of matrix A on several vectors and verify that the action of this matrix on the plane is a rotation counterclockwise by and angle θ, as shown in Fig. 4. A particular case of this matrix was shown in Example 8.2.2, where $\theta = \pi/2$. Since eigenvectors of a matrix determine directions which are left invariant by the action of the matrix, and a rotation does not have such directions, we conclude that the matrix A above does not have eigenvectors and so it does not have eigenvalues either. $\triangleleft$

Remark: We will show that matrix A in Example 8.3.3 has complex-valued eigenvalues.

We now describe a method to find eigenvalue-eigenvector pairs of a matrix, if they exit. In other words, we are going to solve the eigenvalue-eigenvector problem: Given an $n \times n$ matrix A find, if possible, all its eigenvalues and eigenvectors, that is, all pairs λ and $\boldsymbol{v} \neq \boldsymbol{0}$ solutions of the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

This problem is more complicated than finding the solution $\boldsymbol{x}$ to a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A and $\boldsymbol{b}$ are known. In the eigenvalue-eigenvector problem above neither λ nor $\boldsymbol{v}$ are known. To solve the eigenvalue-eigenvector problem for a matrix A we proceed as follows:

(a) First, find the eigenvalues λ;
(b) Second, for each eigenvalue λ, find the corresponding eigenvectors $\boldsymbol{v}$.

The following result summarizes a way to solve the steps above.

Theorem 8.3.2 (Eigenvalues-Eigenvectors)**.**

(a) All the eigenvalues λ of an $n \times n$ matrix A are the solutions of

$$\det(A - \lambda I) = 0. \tag{8.3.1}$$

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the nonzero solutions to the homogeneous linear system

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}. \tag{8.3.2}$$

Proof of Theorem 8.3.2: The number λ and the nonzero vector $\boldsymbol{v}$ are an eigenvalue-eigenvector pair of matrix A iff holds

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \quad \Leftrightarrow \quad (A - \lambda I)\boldsymbol{v} = \boldsymbol{0},$$

where I is the $n \times n$ identity matrix. Since $\boldsymbol{v} \neq \boldsymbol{0}$, the last equation above says that the columns of the matrix $(A - \lambda I)$ are linearly dependent. This last property is equivalent, by Theorem 8.2.12, to the equation

$$\det(A - \lambda I) = 0,$$

which is the equation that determines the eigenvalues λ. Once this equation is solved, substitute each solution λ back into the original eigenvalue-eigenvector equation

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}.$$

Since λ is known, this is a linear homogeneous system for the eigenvector components. It always has nonzero solutions, since λ is precisely the number that makes the coefficient matrix $(A - \lambda I)$ not invertible. This establishes the Theorem. $\square$

Example 8.3.4. Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We first find the eigenvalues as the solutions of the Eq. (8.3.1). Compute

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$0 = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 4, \\ \lambda_- = -2. \end{cases}$$

We have obtained two eigenvalues, so now we introduce $\lambda_+ = 4$ into Eq. (8.3.2), that is,

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Then we solve for $\boldsymbol{v}^+$ the equation

$$(A - 4I)\boldsymbol{v}^+ = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\boldsymbol{v}^+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^+ \quad \Rightarrow \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^+ = 1$. A similar calculation provides the eigenvector $\boldsymbol{v}^-$ associated with the eigenvalue $\lambda_- = -2$, that is, first compute the matrix

$$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

then we solve for $\boldsymbol{v}^-$ the equation

$$(A + 2I)\boldsymbol{v}^- = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^- = -v_2^-, \\ v_2^- \quad \text{free.} \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^- = \begin{bmatrix} -v_2^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^- \quad \Rightarrow \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^- = 1$. We therefore conclude that the eigenvalues and eigenvectors of the matrix A above are given by

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

◁

It is useful to introduce few more concepts, that are common in the literature.

Definition 8.3.3. *The **characteristic polynomial** of an $n \times n$ matrix A is the function*

$$p(\lambda) = \det(A - \lambda I).$$

Example 8.3.5. Find the characteristic polynomial of matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We need to compute the determinant

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda + 1 - 9.$$

We conclude that the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda - 8$. ◁

Since the matrix A in this example is 2×2, its characteristic polynomial has degree two. One can show that the characteristic polynomial of an $n \times n$ matrix has degree n. The eigenvalues of the matrix are the roots of the characteristic polynomial. Different matrices may have different types of roots, so we try to classify these roots in the following definition.

Definition 8.3.4. *Given an $n\times n$ matrix A with real eigenvalues λ_i, where $i = 1, \cdots, k \leqslant n$, it is always possible to express the characteristic polynomial of A as*

$$p(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

The number r_i is called the ***algebraic multiplicity*** *of the eigenvalue λ_i. Furthermore, the* ***geometric multiplicity*** *of an eigenvalue λ_i, denoted as s_i, is the maximum number of eigenvectors of λ_i that form a linearly independent set.*

Example 8.3.6. Find the eigenvalues algebraic and geometric multiplicities of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Solution: In order to find the algebraic multiplicity of the eigenvalues we need first to find the eigenvalues. We now that the characteristic polynomial of this matrix is given by

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9.$$

The roots of this polynomial are $\lambda_1 = 4$ and $\lambda_2 = -2$, so we know that $p(\lambda)$ can be rewritten in the following way,

$$p(\lambda) = (\lambda - 4)(\lambda + 2).$$

We conclude that the algebraic multiplicity of the eigenvalues are both one, that is,

$$\lambda_1 = 4, \quad r_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad r_2 = 1.$$

In order to find the geometric multiplicities of matrix eigenvalues we need first to find the matrix eigenvectors. This part of the work was already done in the Example 8.3.4 above and the result is

$$\lambda_1 = 4, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

From this expression we conclude that the geometric multiplicities for each eigenvalue are just one, that is,

$$\lambda_1 = 4, \quad s_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad s_2 = 1.$$

◁

The following example shows that two matrices can have the same eigenvalues, and so the same algebraic multiplicities, but different eigenvectors with different geometric multiplicities.

Example 8.3.7. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: We start finding the eigenvalues, the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = -\dfrac{v_3^{(1)}}{2}, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free.} \end{cases}$$

Therefore, choosing $v_3^{(1)} = 2$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$, which are all solutions of the linear system

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free,} \\ v_2^{(2)} \quad \text{free,} \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain two linearly independent solutions, the first one $\boldsymbol{v}^{(2)}$ with the choice $v_1^{(2)} = 1$, $v_2^{(2)} = 0$, and the second one $\boldsymbol{w}^{(2)}$ with the choice $v_1^{(2)} = 0$, $v_2^{(2)} = 1$, that is

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 2.$$

Summarizing, the matrix in this example has three linearly independent eigenvectors. ◁

Example 8.3.8. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: Notice that this matrix has only the coefficient a_{12} different from the previous example. Again, we start finding the eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

So this matrix has the same eigenvalues and algebraic multiplicities as the matrix in the previous example. We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = 0, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free.} \end{cases}$$

Therefore, choosing $v_3^{(1)} = 1$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$. However, in this case we obtain only one solution, as this calculation shows,

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free,} \\ v_2^{(2)} = 0, \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain only one linearly independent solution, which corresponds to the choice $v_1^{(2)} = 1$, that is,

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 1.$$

Summarizing, the matrix in this example has only two linearly independent eigenvectors, and in the case of the eigenvalue $\lambda_2 = 3$ we have the strict inequality

$$1 = s_2 < r_2 = 2.$$

$\lhd$

8.3.2. Diagonalizable Matrices. We first introduce the notion of a diagonal matrix. Later on we define a diagonalizable matrix as a matrix that can be transformed into a diagonal matrix by a simple transformation.

Definition 8.3.5. *An $n \times n$ matrix A is called* ***diagonal*** *iff* $A = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}$.

That is, a matrix is diagonal iff every nondiagonal coefficient vanishes. From now on we use the following notation for a diagonal matrix A:

$$A = \operatorname{diag}\big[a_{11}, \cdots, a_{nn}\big] = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}.$$

This notation says that the matrix is diagonal and shows only the diagonal coefficients, since any other coefficient vanishes. The next result says that the eigenvalues of a diagonal matrix are the matrix diagonal elements, and it gives the corresponding eigenvectors.

Theorem 8.3.6. *If $D = \operatorname{diag}[d_{11}, \cdots, d_{nn}]$, then eigenpairs of D are*

$$\lambda_1 = d_{11}, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \lambda_n = d_{nn}, \quad \boldsymbol{v}^{(n)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Diagonal matrices are simple to manipulate since they share many properties with numbers. For example the product of two diagonal matrices is commutative. It is simple to compute power functions of a diagonal matrix. It is also simple to compute more involved functions of a diagonal matrix, like the exponential function.

Example 8.3.9. For every positive integer n find A^n, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution: We start computing A^2 as follows,

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}.$$

We now compute A^3,

$$A^3 = A^2 A = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix}.$$

Using induction, it is simple to see that $A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$. ◁

Many properties of diagonal matrices are shared by diagonalizable matrices. These are matrices that can be transformed into a diagonal matrix by a simple transformation.

Definition 8.3.7. *An $n \times n$ matrix A is called* ***diagonalizable*** *iff there exists an invertible matrix P and a diagonal matrix D such that*

$$A = PDP^{-1}.$$

Remarks:

(a) Systems of linear differential equations are simple to solve in the case that the coefficient matrix is diagonalizable. One decouples the differential equations, solves the decoupled equations, and transforms the solutions back to the original unknowns.

(b) Not every square matrix is diagonalizable. For example, matrix A below is diagonalizable while B is not,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}.$$

Example 8.3.10. Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable, where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Solution: That matrix P is invertible can be verified by computing its determinant, $\det(P) = 1 - (-1) = 2$. Since the determinant is nonzero, P is invertible. Using linear algebra methods one can find out that the inverse matrix is $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Now we only need to verify that PDP^{-1} is indeed A. A straightforward calculation shows

$$\begin{aligned} PDP^{-1} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad PDP^{-1} = A. \end{aligned}$$

$\triangleleft$

There is a deep relation between the eigenpairs of a matrix and whether that matrix is diagonalizable.

Theorem 8.3.8 (Diagonalizable Matrix)**.** *An $n \times n$ matrix A is diagonalizable iff A has a linearly independent set of n eigenvectors. Furthermore, if $\lambda_i, \boldsymbol{v}_i$, for $i = 1, \cdots, n$, are eigenpairs of A, then*

$$A = PDP^{-1}, \quad P = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n], \quad D = \operatorname{diag}\left[\lambda_1, \cdots, \lambda_n\right].$$

Proof of Theorem 8.3.8:

($\Rightarrow$) Since matrix A is diagonalizable, there exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Multiply this equation by P^{-1} on the left and by P on the right, we get

$$D = P^{-1}AP. \tag{8.3.3}$$

Since $n \times n$ matrix D is diagonal, it has a linearly independent set of n eigenvectors, given by the column vectors of the identity matrix, that is,

$$D\boldsymbol{e}^{(i)} = d_{ii}\,\boldsymbol{e}^{(i)}, \qquad D = \operatorname{diag}\left[d_{11}, \cdots, d_{nn}\right], \qquad I = \left[\boldsymbol{e}^{(1)}, \cdots, \boldsymbol{e}^{(n)}\right].$$

So, the pair $d_{ii}, \boldsymbol{e}^{(i)}$ is an eigenvalue-eigenvector pair of D, for $i = 1 \cdots, n$. Using this information in Eq. (8.3.3) we get

$$d_{ii}\,\boldsymbol{e}^{(i)} = D\boldsymbol{e}^{(i)} = P^{-1}AP\boldsymbol{e}^{(i)} \quad \Rightarrow \quad A\big(P\boldsymbol{e}^{(i)}\big) = d_{ii}\,\big(P\boldsymbol{e}^{(i)}\big),$$

where the last equation comes from multiplying the former equation by P on the left. This last equation says that the vectors $\boldsymbol{v}^{(i)} = P\boldsymbol{e}^{(i)}$ are eigenvectors of A with eigenvalue d_{ii}. By definition, $\boldsymbol{v}^{(i)}$ is the i-th column of matrix P, that is,

$$P = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big]\,.$$

Since matrix P is invertible, the eigenvectors set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. This establishes this part of the Theorem.

($\Leftarrow$) Let $\lambda_i, \boldsymbol{v}^{(i)}$ be eigenvalue-eigenvector pairs of matrix A, for $i = 1, \cdots, n$. Now use the eigenvectors to construct matrix $P = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big]$. This matrix is invertible, since the eigenvector set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. We now show that matrix $P^{-1}AP$ is diagonal. We start computing the product

$$AP = A\,\big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big] = \big[A\boldsymbol{v}^{(1)}, \cdots, A\boldsymbol{v}^{(n)}\big]\,, = \big[\lambda_1\boldsymbol{v}^{(1)} \cdots, \lambda_n\boldsymbol{v}^{(n)}\big]\,.$$

that is,

$$P^{-1}AP = P^{-1}\,\big[\lambda_1\boldsymbol{v}^{(1)}, \cdots, \lambda_n\boldsymbol{v}^{(n)}\big] = \big[\lambda_1 P^{-1}\boldsymbol{v}^{(1)}, \cdots, \lambda_n P^{-1}\boldsymbol{v}^{(n)}\big]\,.$$

At this point it is useful to recall that P^{-1} is the inverse of P,

$$I = P^{-1}P \quad \Leftrightarrow \quad \big[\boldsymbol{e}^{(1)}, \cdots, \boldsymbol{e}^{(n)}\big] = P^{-1}\,\big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big] = \big[P^{-1}\boldsymbol{v}^{(1)}, \cdots, P^{-1}\boldsymbol{v}^{(n)}\big]\,.$$

So, $\boldsymbol{e}^{(i)} = P^{-1}\boldsymbol{v}^{(i)}$, for $i = 1 \cdots, n$. Using these equations in the equation for $P^{-1}AP$,

$$P^{-1}AP = \big[\lambda_1\boldsymbol{e}^{(1)}, \cdots, \lambda_n\boldsymbol{e}^{(n)}\big] = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big]\,.$$

Denoting $D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big]$ we conclude that $P^{-1}AP = D$, or equivalently

$$A = PDP^{-1}, \qquad P = \big[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\big]\,, \qquad D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big]\,.$$

This means that A is diagonalizable. This establishes the Theorem. □

Example 8.3.11. Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We know that the eigenvalue-eigenvector pairs are

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

We must show that $A = PDP^{-1}$. This is indeed the case, since

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We conclude, $PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad PDP^{-1} = A$, that is, A is diagonalizable. ◁

With Theorem 8.3.8 we can show that a matrix *is not* diagonalizable.

Example 8.3.12. Show that matrix $B = \dfrac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

Solution: We first compute the matrix eigenvalues. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(\dfrac{3}{2} - \lambda\right) & \dfrac{1}{2} \\ -\dfrac{1}{2} & \left(\dfrac{5}{2} - \lambda\right) \end{vmatrix} = \left(\frac{3}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) + \frac{1}{4} = \lambda^2 - 4\lambda + 4.$$

The roots of the characteristic polynomial are computed in the usual way,

$$\lambda = \frac{1}{2}\big[4 \pm \sqrt{16 - 16}\big] \quad \Rightarrow \quad \lambda = 2, \qquad r = 2.$$

We have obtained a single eigenvalue with algebraic multiplicity $r = 2$. The associated eigenvectors are computed as the solutions to the equation $(A - 2I)\boldsymbol{v} = \boldsymbol{0}$. Then,

$$(A - 2I) = \begin{bmatrix} \left(\dfrac{3}{2} - 2\right) & \dfrac{1}{2} \\ -\dfrac{1}{2} & \left(\dfrac{5}{2} - 2\right) \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s = 1.$$

We conclude that the biggest linearly independent set of eigenvalues for the 2×2 matrix B contains only one vector, insted of two. Therefore, matrix B is not diagonalizable. $\triangleleft$

Theorem 8.3.8 shows the importance of knowing whether an $n \times n$ matrix has a linearly independent set of n eigenvectors. However, more often than not, there is no simple way to check this property other than to compute all the matrix eigenvectors. But there is a simpler particular case, the case when an $n \times n$ matrix has n *different* eigenvalues. Then, we do not need to compute the eigenvectors. The following result says that such matrix always have a linearly independent set of n eigenvectors, hence, by Theorem 8.3.8, it is diagonalizable.

Theorem 8.3.9 (Different Eigenvalues)**.** *If an $n \times n$ matrix has n different eigenvalues, then this matrix has a linearly independent set of n eigenvectors.*

Proof of Theorem 8.3.9: Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A, all different from each other. Let $\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}$ the corresponding eigenvectors, that is, $A\boldsymbol{v}^{(i)} = \lambda_i \boldsymbol{v}^{(i)}$, with $i = 1, \cdots, n$. We have to show that the set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. We assume that the opposite is true and we obtain a contradiction. Let us assume that the set above is linearly dependent, that is, there are constants $c_1, \cdots, c_n$, not all zero, such that,

$$c_1 \boldsymbol{v}^{(1)} + \cdots + c_n \boldsymbol{v}^{(n)} = \boldsymbol{0}. \tag{8.3.4}$$

Let us name the eigenvalues and eigenvectors such that $c_1 \neq 0$. Now, multiply the equation above by the matrix A, the result is,

$$c_1 \lambda_1 \boldsymbol{v}^{(1)} + \cdots + c_n \lambda_n \boldsymbol{v}^{(n)} = \boldsymbol{0}.$$

Multiply Eq. (8.3.4) by the eigenvalue λ_n, the result is,

$$c_1 \lambda_n \boldsymbol{v}^{(1)} + \cdots + c_n \lambda_n \boldsymbol{v}^{(n)} = \boldsymbol{0}.$$

Subtract the second from the first of the equations above, then the last term on the right-hand sides cancels out, and we obtain,

$$c_1(\lambda_1 - \lambda_n)\boldsymbol{v}^{(1)} + \cdots + c_{n-1}(\lambda_{n-1} - \lambda_n)\boldsymbol{v}^{(n-1)} = \boldsymbol{0}. \tag{8.3.5}$$

Repeat the whole procedure starting with Eq. (8.3.5), that is, multiply this later equation by matrix A and also by λ_{n-1}, then subtract the second from the first, the result is,

$$c_1(\lambda_1-\lambda_n)(\lambda_1-\lambda_{n-1})\boldsymbol{v}^{(1)}+\cdots+c_{n-2}(\lambda_{n-2}-\lambda_n)(\lambda_{n-2}-\lambda_{n-1})\boldsymbol{v}^{(n-2)}=\boldsymbol{0}.$$

Repeat the whole procedure a total of $n-1$ times, in the last step we obtain the equation

$$c_1(\lambda_1-\lambda_n)(\lambda_1-\lambda_{n-1})\cdots(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)\boldsymbol{v}^{(1)}=\boldsymbol{0}.$$

Since all the eigenvalues are different, we conclude that $c_1=0$, however this contradicts our assumption that $c_1\neq 0$. Therefore, the set of n eigenvectors must be linearly independent. This establishes the Theorem. □

Example 8.3.13. Is matrix $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ diagonalizable?

Solution: We compute the matrix eigenvalues, starting with the characteristic polynomial,

$$p(\lambda)=\begin{vmatrix}(1-\lambda)&1\\1&(1-\lambda)\end{vmatrix}=(1-\lambda)^2-1=\lambda^2-2\lambda\quad\Rightarrow\quad p(\lambda)=\lambda(\lambda-2).$$

The roots of the characteristic polynomial are the matrix eigenvalues,

$$\lambda_1=0,\qquad \lambda_2=2.$$

The eigenvalues are different, so by Theorem 8.3.9, matrix A is diagonalizable. ◁

8.3.3. Exercises.

8.3.1.- .

8.3.2.- .

8.4. The Matrix Exponential

When we multiply two square matrices the result is another square matrix. This property allow us to define power functions and polynomials of a square matrix. In this section we go one step further and define the exponential of a square matrix. We will show that the derivative of the exponential function on matrices, as the one defined on real numbers, is proportional to itself.

8.4.1. The Exponential Function.

The exponential function defined on real numbers, $f(x) = e^{ax}$, where a is a constant and $x \in \mathbb{R}$, satisfies $f'(x) = a\,f(x)$. We want to find a function of a square matrix with a similar property. Since the exponential on real numbers can be defined in several equivalent ways, we start with a short review of three of ways to define the exponential e^x.

(a) The exponential function can be defined as a generalization of the power function from the positive integers to the real numbers. One starts with positive integers n, defining

$$e^n = e \cdots e, \quad n\text{-times}.$$

Then one defines $e^0 = 1$, and for negative integers $-n$

$$e^{-n} = \frac{1}{e^n}.$$

The next step is to define the exponential for rational numbers, $\frac{m}{n}$, with m, n integers,

$$e^{\frac{m}{n}} = \sqrt[n]{e^m}.$$

The difficult part in this definition of the exponential is the generalization to irrational numbers, x, which is done by a limit,

$$e^x = \lim_{\frac{m}{n} \to x} e^{\frac{m}{n}}.$$

It is nontrivial to define that limit precisely, which is why many calculus textbooks do not show it. Anyway, it is not clear how to generalize this definition from real numbers x to square matrices X.

(b) The exponential function can be defined as the inverse of the natural logarithm function $g(x) = \ln(x)$, which in turns is defined as the area under the graph of the function $h(x) = \frac{1}{x}$ from 1 to x, that is,

$$\ln(x) = \int_1^x \frac{1}{y}\,dy, \qquad x \in (0, \infty).$$

Again, it is not clear how to extend to matrices this definition of the exponential function on real numbers.

(c) The exponential function can be defined also by its Taylor series expansion,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Most calculus textbooks show this series expansion, a Taylor expansion, as a result from the exponential definition, not as a definition itself. But one can define the exponential using this series and prove that the function so defined satisfies the properties in (a) and (b). It turns out, this series expression can be generalized square matrices.

We now use the idea in (c) to define the exponential function on square matrices. We start with the power function of a square matrix, $f(X) = X^n = X \cdots X$, n-times, for X a square matrix and n a positive integer. Then we define a polynomial of a square matrix,

$$p(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 I.$$

Now we are ready to define the exponential of a square matrix.

Definition 8.4.1. *The* ***exponential*** *of a square matrix A is the infinite sum*

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{8.4.1}$$

This definition makes sense, because the infinite sum in Eq. (8.4.1) converges.

Theorem 8.4.2. *The infinite sum in Eq. (8.4.1) converges for all $n \times n$ matrices.*

Proof: See Section 2.1.2 and 4.5 in Hassani [6] for a proof using the Spectral Theorem. □

The infinite sum in the definition of the exponential of a matrix is in general difficult to compute. However, when the matrix is diagonal, the exponential is remarkably simple.

Theorem 8.4.3 (Exponential of Diagonal Matrices)**.** *If $D = \operatorname{diag}\big[d_1, \cdots, d_n\big]$, then*

$$e^D = \operatorname{diag}\big[e^{d_1}, \cdots, e^{d_n}\big].$$

Proof of Theorem 8.4.3: We start from the definition of the exponential,

$$e^D = \sum_{k=0}^{\infty} \frac{1}{k!}\big(\operatorname{diag}\big[d_1, \cdots, d_n\big]\big)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\operatorname{diag}\big[(d_1)^k, \cdots, (d_n)^k\big].$$

where in the second equallity we used that the matrix D is diagonal. Then,

$$e^D = \sum_{k=0}^{\infty} \operatorname{diag}\Big[\frac{(d_1)^k}{k!}, \cdots, \frac{(d_n)^k}{k!}\Big] = \operatorname{diag}\Big[\sum_{k=0}^{\infty}\frac{(d_1)^k}{k!}, \cdots, \sum_{k=0}^{\infty}\frac{(d_n)^k}{k!}\Big].$$

Each sum in the diagonal of matrix above satisfies $\displaystyle\sum_{k=0}^{\infty}\frac{(d_i)^k}{k!} = e^{d_i}$. Therefore, we arrive to the equation $e^D = \operatorname{diag}\big[e^{d_1}, \cdots, e^{d_n}\big]$. This establishes the Theorem. □

Example 8.4.1. Compute e^A, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

Solution: We follow the proof of Theorem 8.4.3 to get this result. We start with the definition of the exponential

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}^n.$$

Since the matrix A is diagonal, we have that

$$\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}^n = \begin{bmatrix} 2^n & 0 \\ 0 & 7^n \end{bmatrix}.$$

Therefore,

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 2^n & 0 \\ 0 & 7^n \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{2^n}{n!} & 0 \\ 0 & \frac{7^n}{n!} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{2^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{7^n}{n!} \end{bmatrix}.$$

Since $\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$, for $a = 2, 7$, we obtain that $e^{\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}} = \begin{bmatrix} e^2 & 0 \\ 0 & e^7 \end{bmatrix}$. $\triangleleft$

8.4.2. Diagonalizable Matrices Formula. The exponential of a diagonalizable matrix is simple to compute, although not as simple as for diagonal matrices. The infinite sum in the exponential of a diagonalizable matrix reduces to a product of three matrices. We start with the following result, the nth-power of a diagonalizable matrix.

Theorem 8.4.4 (Powers of Diagonalizable Matrices)**.** *If an $n \times n$ matrix A is diagonalizable, with invertible matrix P and diagonal matrix D satisfying $A = PDP^{-1}$, then for every integer $k \geqslant 1$ holds*

$$A^k = PD^kP^{-1}. \tag{8.4.2}$$

Proof of Theorem 8.4.4: For any diagonal matrix $D = \operatorname{diag}\big[d_1, \cdots, d_n\big]$ we know that

$$D^n = \operatorname{diag}\big[d_1^n, \cdots, d_n^n\big].$$

We use this result and induction in n to prove Eq.(8.4.2). Since the case $n = 1$ is trivially true, we start computing the case $n = 2$. We get

$$A^2 = \big(PDP^{-1}\big)^2 = \big(PDP^{-1}\big)\big(PDP^{-1}\big) = PDDP^{-1} \quad \Rightarrow \quad A^2 = PD^2P^{-1},$$

that is, Eq. (8.4.2) holds for $k = 2$. Now assume that Eq. (8.4.2) is true for k. This equation also holds for $k + 1$, since

$$A^{(k+1)} = A^kA = \big(PD^kP^{-1}\big)\big(PDP^{-1}\big) = PD^kP^{-1}PDP^{-1} = PD^kDP^{-1}.$$

We conclude that $A^{(k+1)} = PD^{(k+1)}P^{-1}$. This establishes the Theorem. $\square$

We are ready to compute the exponential of a diagonalizable matrix.

Theorem 8.4.5 (Exponential of Diagonalizable Matrices)**.** *If an $n \times n$ matrix A is diagonalizable, with invertible matrix P and diagonal matrix D satisfying $A = PDP^{-1}$, then the exponential of matrix A is given by*

$$e^A = Pe^DP^{-1}. \tag{8.4.3}$$

Remark: Theorem 8.4.5 says that the infinite sum in the definition of e^A reduces to a product of three matrices when the matrix A is diagonalizable. This Theorem also says that *to compute the exponential of a diagonalizable matrix we need to compute the eigenvalues and eigenvectors of that matrix.*

Proof of Theorem 8.4.5: We start with the definition of the exponential,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^n = \sum_{k=0}^{\infty} \frac{1}{k!} (PDP^{-1})^n = \sum_{k=0}^{\infty} \frac{1}{k!} (PD^nP^{-1}),$$

where the last step comes from Theorem 8.4.4. Now, in the expression on the far right we can take common factor P on the left and P^{-1} on the right, that is,

$$e^A = P\Big(\sum_{k=0}^{\infty} \frac{1}{k!} D^n\Big) P^{-1}.$$

The sum in between parenthesis is the exponential of the diagonal matrix D, which we computed in Theorem 8.4.3,

$$e^A = Pe^D P^{-1}.$$

This establishes the Theorem. $\square$

We have defined the exponential function $\tilde{F}(A) = e^A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, which is a function from the space of square matrices into the space of square matrices. However, when one studies solutions to linear systems of differential equations, one needs a slightly different type of functions. One needs functions of the form $F(t) = e^{At} : \mathbb{R} \to \mathbb{R}^{n\times n}$, where A is a constant square matrix and the independent variable is $t \in \mathbb{R}$. That is, one needs to generalize the real constant a in the function $f(t) = e^{at}$ to an $n \times n$ matrix A. In the case that the matrix A is diagonalizable, with $A = PDP^{-1}$, so is matrix At, and $At = P(Dt)P^{-1}$. Therefore, the formula for the exponential of At is simply

$$e^{At} = Pe^{Dt}P^{-1}.$$

We use this formula in the following example.

Example 8.4.2. Compute e^{At}, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $t \in \mathbb{R}$.

Solution: To compute e^{At} we need the decomposition $A = PDP^{-1}$, which in turns implies that $At = P(Dt)P^{-1}$. Matrices P and D are constructed with the eigenvectors and eigenvalues of matrix A. We computed them in Example **??**,

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then, the exponential function is given by

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Usually one leaves the function in this form. If we multiply the three matrices out we get

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

$\lhd$

8.4.3. Properties of the Exponential.

We summarize some simple properties of the exponential function in the following result. We leave the proof as an exercise.

Theorem 8.4.6 (Algebraic Properties)**.** *If A is an $n \times n$ matrix, then*

(a) If 0 is the $n \times n$ zero matrix, then $e^0 = I$.
(b) $\left(e^A\right)^T = e^{(A^T)}$, where T means transpose.
(c) For all nonnegative integers k holds $A^k e^A = e^A A^k$.
(d) If $AB = BA$, then $A\, e^B = e^B A$ and $e^A e^B = e^B e^A$.

An important property of the exponential on real numbers is not true for the exponential on matrices. We know that $e^a e^b = e^{a+b}$ for all real numbers a, b. However, there exist $n\times n$ matrices A, B such that $e^A e^B \neq e^{A+B}$. We now prove a weaker property.

Theorem 8.4.7 (Group Property)**.** *If A is an $n \times n$ matrix and s, t are real numbers, then*

$$e^{As}\, e^{At} = e^{A(s+t)}.$$

Proof of Theorem 8.4.7: We start with the definition of the exponential function

$$e^{As}\, e^{At} = \Big(\sum_{j=0}^{\infty} \frac{A^j\, s^j}{j!}\Big) \Big(\sum_{k=0}^{\infty} \frac{A^k\, t^k}{k!}\Big) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{A^{j+k}\, s^j\, t^k}{j!\, k!}.$$

We now introduce the new label $n = j + k$, then $j = n - k$, and we reorder the terms,

$$e^{As}\, e^{At} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{A^n\, s^{n-k}\, t^k}{(n-k)!\, k!} = \sum_{n=0}^{\infty} \frac{A^n}{n!} \Big(\sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!}\, s^{n-k}\, t^k\Big).$$

If we recall the binomial theorem, $(s+t)^n = \displaystyle\sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!}\, s^{n-k}\, t^k$, we get

$$e^{As}\, e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!}\, (s+t)^n = e^{A(s+t)}.$$

This establishes the Theorem. □

If we set $s = 1$ and $t = -1$ in the Theorem 8.4.7 we get that

$$e^{A}\, e^{-A} = e^{A(1-1)} = e^0 = I,$$

so we have a formula for the inverse of the exponential.

Theorem 8.4.8 (Inverse Exponential)**.** *If A is an $n \times n$ matrix, then*

$$\big(e^{A}\big)^{-1} = e^{-A}.$$

Example 8.4.3. Verify Theorem 8.4.8 for e^{At}, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $t \in \mathbb{R}$.

Solution: In Example 8.4.2 we found that

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

A 2×2 matrix is invertible iff its determinant is nonzero. In our case,

$$\det\big(e^{At}\big) = \frac{1}{2}\,(e^{4t} + e^{-2t})\,\frac{1}{2}\,(e^{4t} + e^{-2t}) - \frac{1}{2}\,(e^{4t} - e^{-2t})\,\frac{1}{2}\,(e^{4t} - e^{-2t}) = e^{2t},$$

hence e^{At} is invertible. The inverse is

$$\big(e^{At}\big)^{-1} = \frac{1}{e^{2t}}\,\frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (-e^{4t} + e^{-2t}) \\ (-e^{4t} + e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix},$$

that is

$$\big(e^{At}\big)^{-1} = \frac{1}{2} \begin{bmatrix} (e^{2t} + e^{-4t}) & (-e^{2t} + e^{-4t}) \\ (-e^{2t} + e^{-4t}) & (e^{2t} + e^{-4t}) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (e^{-4t} + e^{2t}) & (e^{-4t} - e^{2t}) \\ (e^{-4t} - e^{2t}) & (e^{-4t} + e^{2t}) \end{bmatrix} = e^{-At}.$$

◁

We now want to compute the derivative of the function $F(t) = e^{At}$, where A is a constant $n \times n$ matrix and $t \in \mathbb{R}$. It is not difficult to show the following result.

Theorem 8.4.9 (Derivative of the Exponential)**.** *If A is an $n \times n$ matrix, and $t \in \mathbb{R}$, then*

$$\frac{d}{dt} e^{At} = A\, e^{At}.$$

Remark: Recall that Theorem 8.4.6 says that $A\, e^{A} = e^{A}\, A$, so we have that

$$\frac{d}{dt} e^{At} = A\, e^{At} = e^{A}\, A.$$

First Proof of Theorem 8.4.9: We use the definition of the exponential,

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{A^n}{n!} \frac{d}{dt}(t^n) = \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} = A \sum_{n=1}^{\infty} \frac{A^{(n-1)} t^{n-1}}{(n-1)!},$$

therefore we get

$$\frac{d}{dt} e^{At} = A\, e^{At}.$$

This establishes the Theorem. $\square$

Second Proof of Theorem 8.4.9: We use the definition of derivative and Theorem 8.4.7,

$$F'(t) = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h} = \lim_{h \to 0} \frac{e^{At}\, e^{Ah} - e^{At}}{h} = e^{At} \Big(\lim_{h \to 0} \frac{e^{Ah} - I}{h} \Big),$$

and using now the power series definition of the exponential we get

$$F'(t) = e^{At} \Big[\lim_{h \to 0} \frac{1}{h} \Big(Ah + \frac{A^2 h^2}{2!} + \cdots \Big) \Big] = e^{At}\, A.$$

This establishes the Theorem. $\square$

Example 8.4.4. Verify Theorem 8.4.9 for $F(t) = e^{At}$, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $t \in \mathbb{R}$.

Solution: In Example 8.4.2 we found that

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

Therefore, if we derivate component by component we get

$$\frac{d}{dt} e^{At} = \frac{1}{2} \begin{bmatrix} (4\, e^{4t} - 2\, e^{-2t}) & (4\, e^{4t} + 2\, e^{-2t}) \\ (4\, e^{4t} + 2\, e^{-2t}) & (4\, e^{4t} - 2\, e^{-2t}) \end{bmatrix}.$$

On the other hand, if we compute

$$\begin{aligned} A\, e^{At} &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (4\, e^{4t} - 2\, e^{-2t}) & (4\, e^{4t} + 2\, e^{-2t}) \\ (4\, e^{4t} + 2\, e^{-2t}) & (4\, e^{4t} - 2\, e^{-2t}) \end{bmatrix} \end{aligned}$$

Therefore, $\frac{d}{dt} e^{At} = A\, e^{At}$. The relation $\frac{d}{dt} e^{At} = e^{At}\, A$ is shown in a similar way. $\lhd$

We end this review of the matrix exponential showing when formula $e^{A+B} = e^{A}\, e^{B}$ holds.

Theorem 8.4.10 (Exponent Rule)**.** *If A, B are $n \times n$ matrices such that $AB = BA$, then*

$$e^{A+B} = e^{A}\, e^{B}.$$

Proof of Theorem 8.4.10: Introduce the function

$$F(t) = e^{(A+B)t}\, e^{-Bt}\, e^{-At},$$

where $t \in \mathbb{R}$. Compute the derivative of F,

$$F'(t) = (A+B)\, e^{(A+B)t}\, e^{-Bt}\, e^{-At} + e^{(A+B)t}\, (-B)\, e^{-Bt}\, e^{-At} + e^{(A+B)t}\, e^{-Bt}\, (-A)\, e^{-At}.$$

Since $AB = BA$, we know that $e^{-BT} A = A\, e^{-Bt}$, so we get

$$F'(t) = (A+B)\, e^{(A+B)t}\, e^{-Bt}\, e^{-At} - e^{(A+B)t}\, B\, e^{-Bt}\, e^{-At} - e^{(A+B)t}\, A\, e^{-Bt}\, e^{-At}.$$

Now $AB = BA$ also implies that $(A+B)\, B = B\, (A+B)$, therefore Theorem 8.4.6 implies

$$e^{(A+B)t}\, B = B\, e^{(A+B)t}.$$

Analogously, we have that $(A+B)\, A = A\, (A+B)$, therefore Theorem 8.4.6 implies that

$$e^{(A+B)t}\, A = A\, e^{(A+B)t}.$$

Using these equation in F' we get

$$F'(t) = (A+B)\, F(t) - B\, F(t) - A\, F(t) \quad \Rightarrow \quad F'(t) = 0.$$

Therefore, $F(t)$ is a constant matrix, $F(t) = F(0) = I$. So we get

$$e^{(A+B)t}\, e^{-Bt}\, e^{-At} = I \quad \Rightarrow \quad e^{(A+B)t} = e^{At}\, e^{Bt}.$$

This establishes the Theorem. $\square$

8.4.4. Exercises.

8.4.1.- Use the definition of the matrix exponential to prove Theorem 8.4.6. Do not use any other theorems in this Section.

8.4.2.- Compute e^A for the matrices

(a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$;

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

8.4.3.- Show that, if A is diagonalizable,

$$\det\; e^{A}) = e^{\operatorname{tr}(A)}.$$

Remark: This result is true for all square matrices, but it is hard to prove for nondiagonalizable matrices.

8.4.4.- A square matrix P is a projection iff

$$P^2 = P.$$

Compute the exponential of a projection matrix, e^P. Your answer must not contain any infinite sum.

8.4.5.- If $A^2 = I$, show that

$$2\,e^A = \Big(e + \frac{1}{e}\Big)\, I + \Big(e - \frac{1}{e}\Big)\, A.$$

8.4.6.- If λ and $\boldsymbol{v}$ are an eigenvalue and eigenvector of A, then show that

$$e^A \boldsymbol{v} = e^{\lambda}\, \boldsymbol{v}.$$

8.4.7.- Compute e^A for the matrices

(a) $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$;

(b) $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$.

8.4.8.- * Compute the function $F(t) = e^{At}$ for any real number t and

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.$$

8.4.9.- Compute the function $F(t) = e^{At}$ for any real number t and

$$A = \begin{bmatrix} -7 & 2 \\ -24 & 7 \end{bmatrix}.$$

8.4.10.- By direct computation show that $e^{(A+B)} \neq e^A\, e^B$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

CHAPTER 9

Appendices

A. Overview of Complex Numbers

The first public appearance of complex numbers was in 1545 Gerolamo Cardano's Ars Magna, when he published a way to find solutions of a cubic equation $ax^3 + bx + c = 0$. The solution formula was not his own but given to him sometime earlier by Scipione del Ferro. In order to get such formula there was a step in the calculation involving a $\sqrt{-1}$, which was a mystery for the mathematicians of that time. There is no real number so that its square is -1, so what the heck does this symbol $\sqrt{-1}$ mean? More intriguing, a few steps later during the calculation, this $\sqrt{-1}$ cancels out, and it does not appear in the final formula for the roots of the cubic equation. It was like a ghost entered your calculation and walked out of it without leaving a trace. Maybe we should call them magic numbers.

Everything in nature is magic until we understand how it works, then knowledge advances and magic retreats, one step at a time. It took a while, until the beginning of the 19th century with the—independent but almost simultaneous—works of Karl Gauss and William Hamilton, but our magic numbers were finally understood and they became the complex numbers.

In spite of their name, there is nothing complex about complex numbers. *Planar numbers* is a better fit to what they are—the set of all ordered pairs of real numbers together with specific addition and multiplication rules. Complex numbers can be identified with points on a plane, in the same way that real numbers can be identified with points on a line.

Definition A.1. ***Complex numbers*** *are numbers of the form*

$$(a, b),$$

where a and b are real numbers, together with the operations of addition,

$$(a, b) + (c, d) = (a + c, b + d), \tag{A.1}$$

and multiplication,

$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{A.2}$$

The operation of addition is simple to understand because it is exactly how we add vectors on a plane,

$$\langle a, b\rangle + \langle c, d\rangle = \langle (a + c), (b + d)\rangle.$$

It is the multiplication what distinguishes complex numbers from vectors on the plane. To understand these operations it is useful to start with the following properties.

Theorem A.2. *The addition and multiplication of complex number are commutative, associative, and distributive. That is, given arbitrary complex numbers x, y, and z holds*

(a) Commutativity: $x + y = y + x$ and $x\,y = y\,x$.
(b) Associativity: $x + (y + z) = (x + y) + z$ and $x\,(y\,z) = (x\,y)\,x$.
(c) Distributivity: $x\,(y + z) = x\,y + x\,z$.

Proof of Theorem A.2: We show how to prove one of these properties, the proof for the rest is similar. Let's see the commutativity of multiplication. Given the complex numbers $x = (a, b)$ and $y = (c, d)$ we have

$$x\, y = (a, b)(c, d) = \big((ac - bd), (ad + bc)\big)$$

and

$$y\, x = (c, d)(a, b) = \big((ca - db), (cb + da)\big)$$

therefore we get that $x\, y = y\, x$. The rest of the properties can be proven in a similar way. This establishes the Theorem. □

We now mention a few more properties of complex numbers which are straightforward from the definitions above. For all complex numbers (a, b) we have that

$$\begin{aligned} (0, 0) + (a, b) &= (a, b) \\ (-a, -b) + (a, b) &= (0, 0) \\ (a, b)(1, 0) &= (a, b). \end{aligned}$$

From the first equation above the complex number $(0, 0)$ is called the *zero* complex number. From the second equation above the complex number $(-a, -b)$ is called the *negative* of (a, b), and we write

$$-(a, b) = (-a, -b).$$

From the last equation above the complex number $(1, 0)$ is called the *identity* for the multiplication.

The *inverse* of a complex number (a, b), denoted as $(a, b)^{-1}$, is the complex number satisfying

$$(a, b)\, (a, b)^{-1} = (1, 0).$$

Since the inverse of a complex number is itself a complex number, it can be written as

$$(a, b)^{-1} = (c, d)$$

for appropriate components c and d. The next result gives us a formula for these components. The next result says that every nonzero complex number has an inverse.

Theorem A.3. *The inverse of (a, b), with either $a \neq 0$ or $b \neq 0$, is*

$$(a, b)^{-1} = \Big(\frac{a}{(a^2 + b^2)}, \frac{-b}{(a^2 + b^2)}\Big). \tag{A.3}$$

Proof of Theorem A.3: A complex number $(a, b)^{-1}$ is the inverse of (a, b) iff

$$(a, b)\, (a, b)^{-1} = (1, 0).$$

When we write $(a, b)^{-1} = (c, d)$, the equation above is

$$(a, b)(c, d) = (1, 0).$$

If we compute explicitly the left-hand side above we get

$$\big((ac - bd), (ad + bc)\big) = (1, 0).$$

The equation above implies two equations for real numbers,

$$ac - bd = 1, \qquad ad + bc = 0.$$

In the case that either $a \neq 0$ or $b \neq 0$, the solution to the equations above is

$$c = \frac{a}{(a^2 + b^2)}, \qquad d = \frac{-b}{(a^2 + b^2)}.$$

Therefore, the inverse of (a,b) is

$$(a,b)^{-1} = \Big(\frac{a}{(a^2+b^2)}, \frac{-b}{(a^2+b^2)}\Big).$$

This establishes the Theorem. □

Example A.1. Find the inverse of $(2,3)$. Then, verify your result.

Solution: The formula above says that $(2,3)^{-1}$ is given by

$$(2,3)^{-1} = \Big(\frac{2}{(2^2+3^2)}, \frac{-3}{(2^2+3^2)}\Big) \quad \Rightarrow \quad (2,3)^{-1} = \Big(\frac{2}{13}, \frac{-3}{13}\Big).$$

This is correct, since

$$\begin{aligned}(2,3)\Big(\frac{2}{13}, \frac{-3}{13}\Big) &= \Big(\big(\frac{4}{13} - \frac{(-9)}{13}\big), \big(\frac{-6}{13} + \frac{6}{13}\big)\Big)\\ &= \Big(\frac{13}{13}, \frac{0}{13}\Big)\\ &= (1,0).\end{aligned}$$

◁

A.1. Extending the Real Numbers. The set of all complex numbers of the form $(a,0)$ satisfy the same properties as the set of all real numbers a. Indeed, for all a, c reals holds

$$(a,0) + (c,0) = (a+c,0), \qquad (a,0)(c,0) = (ac,0).$$

We also have that

$$-(a,0) = (-a,0),$$

and the formula above for the inverse of a complex number says that

$$(a,0)^{-1} = \Big(\frac{1}{a}, 0\Big).$$

From here it is natural to identify a complex number $(a,0)$ with the real number a, that is,

$$(a,0) \quad \longleftrightarrow \quad a.$$

This identification suggests the following definition.

Definition A.4. *The **real part** of $z = (a,b)$ is a and—then it is natural to call—the **imaginary part** of z is b. We also use the notation*

$$a = \mathsf{Re}(z), \qquad b = \mathsf{Im}(z).$$

A.2. The Imaginary Unit. We understood complex numbers of the form $(a,0)$. They are no more than the real numbers. Now we study complex numbers of the form $(0,b)$—complex numbers with no real part. In particular, we focus on the complex number $(0,1)$, which we call the *imaginary unit*. Let us compute its square,

$$(0,1)^2 = (0,1)(0,1) = (-1,0) = -(1,0) \quad \Rightarrow \quad (0,1)^2 = -(1,0).$$

Within the complex numbers we do have a number whose square is negative one, and that number is the imaginary unit $(0,1)$. Actually, there are two complex numbers whose square is negative one, one is $(0,1)$ and the other is $-(0,1)$, because

$$(0,-1)^2 = (0,-1)(0,-1) = (0-(-1)(-1), 0+0) = (-1,0) = -(1,0).$$

So, in the set of complex numbers we do have solutions for the $\sqrt{-(1,0)}$, given by

$$\sqrt{-(1,0)} = \pm(0,1).$$

Notice that $\sqrt{-1}$ has no solutions, but $\sqrt{-(1,0)}$ has two solutions. This is the origin of the confusion with del Ferro's calculation. Most of his calculation used numbers of the form $(a,0)$—written as a— except at one tiny spot where a number $(0,1)$ shows up and later on cancels out. Del Ferro's calculation makes perfect sense in the complex realm, and almost all of it can be reproduced with real numbers, but not all.

A.3. Standard Notation. We can now relate the ordered pair notation we have been using for complex numbers with the notation used by the early mathematicians. We start noticing that

$$(a,b) = (a,0) + (0,b) = (a,0) + (b,0)(0,1).$$

Therefore, if we write a for $(a,0)$, b for $(b,0)$, and we use $i = (0,1)$, we get that every complex number (a,b) can be written as

$$(a,b) = a + bi.$$

Recall, a and b are the real and imaginary parts of (a,b). And the equation

$$(0,1)^2 = -(1,0)$$

in the new notation is

$$i^2 = -1.$$

This notation $(a+bi)$ is useful to manipulate formulas involving addition and multiplication. If we multiply $(a+bi)$ by $(c+di)$ and use the distributive and associative properties we get

$$(a+bi)(c+di) = ac + adi + cbi + bdi^2,$$

and if we recall that $i^2 = -1$ and we reorder terms, we get

$$(a+bi)(c+di) = ac - bd + (ad+bc)i.$$

So, we do not need to remember the formula for the product of two complex numbers. With the new notation, this formula comes from the distributive and associative properties. Similarly, to compute the inverse of a complex number $a+bi$ we may write

$$\begin{aligned} \frac{1}{a+bi} &= \frac{1}{(a+bi)}\,\frac{(a-bi)}{(a-bi)} \\ &= \frac{(a-bi)}{(a+bi)(a-bi)}. \end{aligned}$$

Notice that

$$(a+bi)(a-bi) = a^2 + b^2,$$

which has only a real part. Then we can write

$$\frac{1}{a+bi} = \frac{a-bi}{(a^2+b^2)} \quad\Rightarrow\quad \frac{1}{a+bi} = \frac{a}{(a^2+b^2)} - \frac{b}{(a^2+b^2)}\,i$$

which agrees with the formula we got in Theorem A.3.

A.4. Useful Formulas. The powers of i can have only four possible results.

Theorem A.5. *The integer powers of i can have only four results: 1, i, -1, and $-i$.*

Proof of Theorem A.5: We just show that this is the case for the first powers. Dy definition of a power zero and power one we know that

$$i^0 = 1,$$
$$i^1 = i.$$

We also know that

$$i^2 = -i.$$

We can compute the next powers, using that $(a+bi)^{m+n} = (a+bi)^m(a+bi)^n$, so we get

$$\begin{aligned}
i^3 &= i^2\, i = (-1)\, i = -i\\
i^4 &= i^3\, i = -i\, i = -i^2 = 1\\
i^5 &= i^4\, i = (1)\, i = i\\
i^6 &= i^5\, i = i\, i = -1\\
i^7 &= i^6\, i = (-1)\, i = -i\\
&\vdots
\end{aligned}$$

An argument using induction would proof this Theorem. □

The *conjugate* of a complex number $a+bi$ is the complex number

$$\overline{a+bi} = a - bi.$$

For example,

$$\overline{1+2i} = 1-2i, \qquad \overline{a} = a, \qquad \overline{i} = -i, \qquad \overline{4i} = -4i.$$

If we conjugate twice we get the original complex number, that is $\overline{\overline{a+bi}} = a+bi$.

The *modulus* or *absolute value* of a complex number $a+bi$ is the real number

$$|a+bi| = \sqrt{a^2+b^2}.$$

For example

$$|3+4i| = \sqrt{9+16} = \sqrt{25} = 5, \qquad |a+0i| = |a|, \qquad |i| = 1, \qquad |1+i| = \sqrt{2}.$$

Using these definitions is simple to see that

$$(a+bi)\,\overline{(a+bi)} = (a+bi)(a-bi) = (a^2+b^2) = |a+bi|^2.$$

Using these definitions we can rewrite the formula in Eq. (A.3) for the inverse of a complex number as follows,

$$\frac{1}{(a+bi)} = \frac{1}{(a^2+b^2)}(a-bi).$$

If we call $z = a+bi$, then the formula for z^{-1} reduces to

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

Example A.2. Write $\dfrac{1}{(3+4i)}$ in the form $c+di$.

Solution: You multiply numerator and denominator by $3-4i$,

$$\begin{aligned}\frac{1}{(3+4i)} &= \frac{1}{(3+4i)}\,\frac{(3-4i)}{(3-4i)}\\ &= \frac{(3-4i)}{(3^2+4^2)}\\ &= \frac{3-4i}{25}\\ &= \frac{3}{25}-\frac{4}{25}\,i.\end{aligned}$$

So, we have found that the inverse of $(3+4i)$ is $\Big(\dfrac{3}{25}-\dfrac{4}{25}\,i\Big)$. ◁

The absolute value of complex numbers satisfy the triangle inequality.

Theorem A.6. *For all complex numbers z_1, z_2 holds $|z_1+z_2| \leqslant |z_1|+|z_2|$.*

Remark: The idea of the Proof of Theorem A.6 is to use the graphical representations of complex numbers as vectors on a plane. Then $|z_1|$ is the length of the vector given by z_1, and the same holds for the vectors associated to z_2 and z_1+z_2, the latter being the diagonal in the parallelogram formed by z_1 and z_2. Then it is clear that the triangle inequality holds.

The absolute value of a complex number also satisfies the following properties.

Theorem A.7. *For all complex numbers z_1, z_2 holds $|z_1z_2| = |z_1|\,|z_2|$.*

Proof of Theorem A.7: For an arbitrary complex numbers $z_1 = a+bi$ and $z_2 = c+di$, we have

$$z_1z_2 = (ac-bd)+(ad+bc)i,$$

therefore,

$$\begin{aligned}|z_1z_2|^2 &= (ac-bd)^2+(ad+bc)^2\\ &= (ac)^2+(bd)^2-2acbd+(ad)^2+(bc)^2+2adbc\\ &= a^2c^2+b^2d^2+a^2d^2+b^2c^2\\ &= a^2(c^2+d^2)+b^2(d^2+c^2)\\ &= (a^2+b^2)(d^2+c^2)\\ &= |z_1|^2\,|z_2|^2,\end{aligned}$$

Taking a square root we get

$$|z_1z_2| = |z_1|\,|z_2|.$$

This establishes the Theorem. □

Theorem A.8. *Every complex number z satisfies that $|z^n| = |z|^n$, for all integer n.*

Proof of Theorem A.8: One proof uses the previous theorem A.7 and induction in n. For $n = 2$ it is proven by the theorem above,

$$|z^2| = |zz| = |z|\,|z| = |z|^2.$$

Now, suppose the theorem is true for $n-1$, so $|z^{n-1}| = |z|^{n-1}$. Then

$$|z^n| = |z^{n-1}z| = |z^{n-1}|\,|z|$$

where we used the previous theorem A.7. But in the first factor we use the inductive hypothesis,

$$|z^{n-1}|\,|z| = |z|^{n-1}\,|z| = |z|^n.$$

So we have proven that $|z^n| = |z|^n$. This establishes the Theorem. ☐

Remark: A second proof, independent of the previous theorem is that, for an arbitrary non-negative integer n we have,

$$|z^n| = \sqrt{z^n\overline{z^n}} = \sqrt{z^n(\overline{z})^n} = \sqrt{(z\overline{z})^n} = \big(\sqrt{z\overline{z}}\big)^n = |z|^n$$

Example A.3. Verify the result in Theorem A.8 for $n = 3$ and $z = 3 + 4i$.

Solution: First we compute $|z|$ and then its cube,

$$|z| = |3 + 4i| = \sqrt{9 + 16} = 5 \quad \Rightarrow \quad |z|^3 = 125.$$

We now compute z^3, and then its absolute value,

$$z^3 = (3 + 4i)(3 + 4i)(3 + 4i) = -117 + 44\,i \quad \Rightarrow \quad |z^3| = \sqrt{117^2 + 44^2} = 125.$$

Therefore, $|z|^3 = |z^3|$. As an extra bonus, we found another perfect triple, besides the famous $3^2 + 4^2 = 5^2$, which is

$$44^2 + 117^2 = 125^2.$$

◁

A.5. Complex Functions.

We know how to add and multiply complex numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

This means we know how to extend any real-valued function defined on real numbers having a Taylor series expansion. We use the function Taylor series as the definition of the function for complex numbers. For example, the real-valued exponential function has the Taylor series expansion

$$e^{at} = \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}.$$

Therefore, we define the complex-valued exponential as follows.

Definition A.9. *The* ***complex-valued exponential function*** *is given by*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{A.4}$$

Remark: We are particularly interested in the case that the argument of the exponential function is of the form $z = (a \pm bi)t$, where $r_\pm = a \pm bi$ are the roots of the characteristic polynomial of a second order linear differential equation with constant coefficients. In this case, the exponential function has the form

$$e^{(a+bi)t} = \sum_{n=0}^{\infty} \frac{(a+bi)^n t^n}{n!}.$$

The infinite sum on the right-hand side in equation (A.4) makes sense, since we know how to multiply—hence compute powers—of complex numbers, and we know how to add complex numbers. Furthermore, one can prove that the infinite series above converges, because the series converges in absolute value, which implies that the series itself converges. Also important, the name we chose for the function above, the exponential, is well chosen, because this function satisfies the exponential property.

Theorem A.10 (Exp. Property)**.** *For all complex numbers z_1, z_2 holds $e^{z_1+z_2} = e^{z_1}\, e^{z_2}$.*

Proof of Theorem A.10: A straightforward calculation using the binomial formula implies

$$\begin{aligned} e^{z_1+z_2} &= \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{z_1^k\, z_2^{n-k}}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k\, z_2^{n-k}}{k!(n-k)!}, \end{aligned}$$

where we used the notation $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. This double sum is over the triangular region in the nk space given by

$$0 \leqslant n \leqslant \infty \qquad 0 \leqslant k \leqslant n.$$

We now interchange the order of the sums, the indices be given by

$$0 \leqslant k \leqslant \infty \qquad k \leqslant n \leqslant \infty,$$

so we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k\, z_2^{n-k}}{k!(n-k)!} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{z_1^k\, z_2^{n-k}}{k!(n-k)!}.$$

If we introduce the variable $m = n - k$ we get that

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{z_1^k\, z_2^{n-k}}{k!(n-k)!} &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{z_1^k\, z_2^m}{k!m!} \\ &= \Big(\sum_{k=0}^{\infty} \frac{z_1^k}{k!} \Big) \Big(\sum_{m=0}^{\infty} \frac{z_2^m}{m!} \Big) \\ &= e^{z_1}\, e^{z_2}. \end{aligned}$$

So we have shown that $e^{z_1+z_2} = e^{z_1}\, e^{z_2}$. This Establishes the Theorem. □

The exponential property in the case that the exponent is $z = (a+bi)t$ has the form

$$e^{(a+bi)t} = e^{at}\, e^{ibt}.$$

The first factor on the right-hand side above is a real exponential, which—for a given value of $a \neq 0$—it is either a decreasing ($a < 0$) or increasing ($a > 0$) function of t. The second factor above is an exponential of a pure imaginary exponent. These exponentials can be summed in a closed form.

Theorem A.11 (Euler Formula)**.** *For any real number θ holds that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.*

Proof of Theorem A.11: Recall that i^n can have only four results, 1, i, -1, $-i$. This result can be summarized as

$$i^{2n} = (-1)^n \quad \Rightarrow \quad i^{2n+1} = (-1)^n\, i.$$

If we split the sum in the definition of the exponential into even and odd terms in the sum index, we get

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!},$$

and using the property above on the powers of i we get

$$\sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!}.$$

Recall that Taylor series expansions of the sine and cosine functions

$$\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!}, \qquad \cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!}.$$

Therefore, we have shown that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

This establishes the Theorem. □

A.6. Complex Vectors. We can extend the notion of vectors with real components to vectors with complex components. For example, complex-valued vectors on a plane are vectors of the form

$$\boldsymbol{v} = \langle a + bi, c + di \rangle,$$

where a, b, c, d are real numbers. We can add two complex-valued vectors component-wise. So, given

$$\boldsymbol{v}_1 = \langle a_1 + b_1 i, c_1 + d_1 i \rangle, \qquad \boldsymbol{v}_2 = \langle a_2 + b_2 i, c_2 + d_2 i \rangle,$$

we have that

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \big\langle (a_1 + a_2) + (b_1 + b_2)i, (c_1 + c_2) + (d_1 + d_2)i \big\rangle.$$

For example

$$\langle 2 + 3i, 4 + 5i \rangle + \langle 6 + 7i, 8 + 9i \rangle = \langle 8 + 10i, 12 + 14i \rangle.$$

We can also multiply a complex-valued vector by a scalar, which now is a complex number. So, given $\boldsymbol{v} = \langle a + bi, c + di \rangle$ and $z = z_1 + z_2 i$, then

$$z\boldsymbol{v} = (z_1 + z_2 i)\langle a + bi, c + di \rangle = \big\langle (z_1 + z_2 i)(a + bi), (z_1 + z_2 i)(c + di) \big\rangle.$$

For example

$$\begin{aligned} i\,\langle 2 + 3i, 4 + 5i \rangle &= \langle 2i - 3, 4i - 5 \rangle \\ &= \langle -3 + 2i, -5 + 4i \rangle. \end{aligned}$$

The only non-intuitive calculation with complex-valued vectors is how to find the length of a complex vector. Recall that in the case of a real-valued vector $\boldsymbol{v} = \langle a, b\rangle$, the length of the vector is defined as

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{a^2+b^2},$$

where $\cdot$ is the dot product of vectors, that is, given the real-valued vectors $\boldsymbol{v}_1 = \langle a_1, b_1\rangle$, $\boldsymbol{v}_2 = \langle a_2, b_2\rangle$, their dot product is the real number

$$\boldsymbol{v}_1\cdot\boldsymbol{v}_2 = a_1a_2 + b_1b_2.$$

We want to generalize the notion of length from real-valued vectors to complex-valued vectors. Notice that the *length of a vector*—real or complex—must be a *real number*. Unfortunately, in the case of a complex-valued vector $\boldsymbol{v} = \langle a+bi, c+di\rangle$ the formula $\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$ is not always a real number, it may have a nonzero imaginary part. In order to get a real number for the length of a complex-valued vector we define

$$\|\boldsymbol{v}\| = \sqrt{\overline{\boldsymbol{v}}\cdot\boldsymbol{v}},$$

where the conjugate of a vector means to conjugate all its components, that is

$$\overline{\boldsymbol{v}} = \overline{\langle a+bi, c+di\rangle} = \langle a-bi, c-di\rangle.$$

We needed to introduce the conjugate in the first vector in the formula above so that the result is a real number. Indeed, we have the following result.

Theorem A.12. *The length of a complex-valued vector* $\boldsymbol{v} = \langle a+bi, c+di\rangle$ *is*

$$\|\boldsymbol{v}\| = \sqrt{\overline{\boldsymbol{v}}\cdot\boldsymbol{v}} = \sqrt{a^2+b^2+c^2+d^2}.$$

Proof of Theorem A.12: This is a straightforward calculation,

$$\begin{aligned}\|\boldsymbol{v}\|^2 &= \overline{\boldsymbol{v}}\cdot\boldsymbol{v}\\ &= \langle a-bi, c-di\rangle\cdot\langle a+bi, c+di\rangle\\ &= (a-bi)(a+bi) + (c-di)\,(c+di)\\ &= a^2+b^2+c^2+d^2.\end{aligned}$$

So we get the formula

$$\|\boldsymbol{v}\| = \sqrt{a^2+b^2+c^2+d^2}.$$

This establishes the Theorem. □

Example A.4. Find the length of $\boldsymbol{v} = \langle 1+2i, 3+4i\rangle$

Solution: The length of this vector is

$$\|\boldsymbol{v}\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}.$$

◁

A *unit vector* is a vector with length one, that is, $\boldsymbol{u}$ is a unit vector iff $\|\boldsymbol{u}\| = 1$. Sometimes one needs to find a unit vector parallel to some vector $\boldsymbol{v}$. For both real-valued and complex-valued vectors we have the same formula. A unit vector $\boldsymbol{u}$ parallel to $\boldsymbol{v} \neq \boldsymbol{0}$ is

$$\boldsymbol{u} = \frac{1}{\|\boldsymbol{v}\|}\,\boldsymbol{v}.$$

Example A.5. Find a unit vector in the direction of $\boldsymbol{v} = \langle 3+2i, 1-2i \rangle$.

Solution: First we check that $\boldsymbol{v}$ is not a unit vector. Indeed,

$$\begin{aligned}
\|\boldsymbol{v}\|^2 &= \overline{\boldsymbol{v}} \cdot \boldsymbol{v} \\
&= \langle 3-2i, 1+2i \rangle \cdot \langle 3+2i, 1-2i \rangle \\
&= (3-2i)(3+2i) + (1+2i)(1-2i) \\
&= 3^2 + 2^2 + 1^2 + 2^2 \\
&= 14.
\end{aligned}$$

Since $\|\boldsymbol{v}\| = \sqrt{14}$, the vector $\boldsymbol{v}$ is not unit. A unit vector is

$$\boldsymbol{u} = \frac{1}{\sqrt{14}} \langle 3-2i, 1+2i \rangle$$

more explicitly,

$$\boldsymbol{u} = \Big\langle \Big(\frac{3}{\sqrt{14}} - \frac{2}{\sqrt{14}}\, i \Big), \Big(\frac{1}{\sqrt{14}} + \frac{2}{\sqrt{14}}\, i \Big) \Big\rangle$$

$\triangleleft$

Notes.

This appendix is inspired on Tom Apostol's overview of complex numbers given in his outstanding Calculus textbook, [1], Volume I, § 9.

B. Overview of Power Series

We summarize a few results on power series that we will need to find solutions to differential equations. A more detailed presentation of these ideas can be found in standard calculus textbooks, [1, 2, 11, 13]. We start with the definition of analytic functions, which are functions that can be written as a power series expansion on an appropriate domain.

Definition B.1. *A function y is* ***analytic*** *on an interval $(x_0-\rho, x_0+\rho)$ iff it can be written as the power series expansion below, convergent for $|x-x_0|<\rho$,*

$$y(x)=\sum_{n=0}^{\infty} a_n\,(x-x_0)^n.$$

Example B.1. We show a few examples of analytic functions on appropriate domains.

(a) The function $y(x)=\dfrac{1}{1-x}$ is analytic on the interval $(-1,1)$, because it has the power series expansion centered at $x_0=0$, convergent for $|x|<1$,

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots.$$

It is clear that this series diverges for $x \geqslant 1$, but it is not obvious that this series converges if and only if $|x|<1$.

(b) The function $y(x)=e^x$ is analytic on $\mathbb{R}$, and can be written as the power series

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.$$

(c) A function y having at x_0 both infinitely many continuous derivatives and a convergent power series is analytic where the series converges. The Taylor expansion centered at x_0 of such a function is

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}\,(x-x_0)^n,$$

and this means

$$y(x)=y(x_0)+y'(x_0)\,(x-x_0)+\frac{y''(x_0)}{2!}\,(x-x_0)^2+\frac{y'''(x_0)}{3!}\,(x-x_0)^3+\cdots.$$

◁

The Taylor series can be very useful to find the power series expansions of function having infinitely many continuous derivatives.

Example B.2. Find the Taylor series of $y(x)=\sin(x)$ centered at $x_0=0$.

Solution: We need to compute the derivatives of the function y and evaluate these derivatives at the point we center the expansion, in this case $x_0=0$.

$$\begin{aligned} y(x)&=\sin(x) &&\Rightarrow & y(0)&=0, & y'(x)&=\cos(x) &&\Rightarrow & y'(0)&=1,\\ y''(x)&=-\sin(x) &&\Rightarrow & y''(0)&=0, & y'''(x)&=-\cos(x) &&\Rightarrow & y'''(0)&=-1.\end{aligned}$$

One more derivative gives $y^{(4)}(t) = \sin(t)$, so $y^{(4)} = y$, the cycle repeats itself. It is not difficult to see that the Taylor's formula implies,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \Rightarrow \quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{(2n+1)}.$$

◁

Remark: The Taylor series at $x_0 = 0$ for $y(x) = \cos(x)$ is computed in a similar way,

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{(2n)}.$$

Elementary functions like quotient of polynomials, trigonometric functions, exponential and logarithms can be written as power series. But the power series of any of these functions may not be defined on the whole domain of the function. The following example shows a function with this property.

Example B.3. Find the Taylor series for $y(x) = \dfrac{1}{1-x}$ centered at $x_0 = 0$.

Solution: Notice that this function is well defined for every $x \in \mathbb{R} - \{1\}$. The function graph can be seen in Fig. **??**. To find the Taylor series we need to compute the n-derivative, $y^{(n)}(0)$. It simple to check that,

$$y^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad \text{so} \quad y^{(n)}(0) = n!.$$

We conclude that: $y(x) = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$.

One can prove that this power series converges if and only if $|x| < 1$. ◁

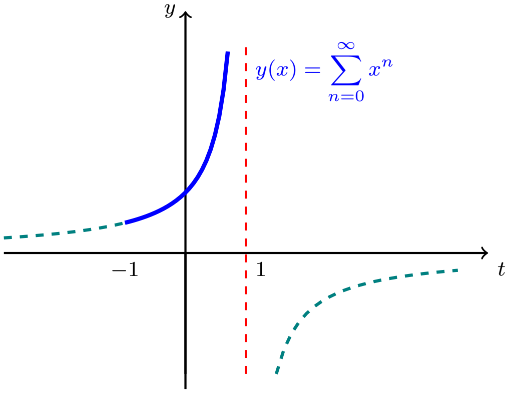


FIGURE 1. The graph of $y = \dfrac{1}{(1-x)}$.

Remark: The power series $y(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ does not converge on $(-\infty, -1] \cup [1, \infty)$. But there are different power series that converge to $y(x) = \dfrac{1}{1-x}$ on intervals inside that domain. For example the Taylor series about $x_0 = 2$ converges for $|x-2| < 1$, that is $1 < x < 3$.

$$y^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \quad \Rightarrow \quad y^{(n)}(2) = \frac{n!}{(-1)^{n+1}} \quad \Rightarrow \quad y(x) = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n.$$

Later on we might need the notion of convergence of an infinite series in absolute value.

Definition B.2. *The power series* $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ ***converges in absolute value*** *iff the series* $\sum_{n=0}^{\infty} |a_n|\,|x - x_0|^n$ *converges.*

Remark: If a series converges in absolute value, it converges. The converse is not true.

Example B.4. One can show that the series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but this series does not converge absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. See [11, 13]. $\triangleleft$

Since power series expansions of functions might not converge on the same domain where the function is defined, it is useful to introduce the region where the power series converges.

Definition B.3. *The* ***radius of convergence*** *of a power series* $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ *is the number* $\rho \geqslant 0$ *satisfying both the series converges absolutely for* $|x - x_0| < \rho$ *and the series diverges for* $|x - x_0| > \rho$.

Remark: The radius of convergence defines the size of the biggest open interval where the power series converges. This interval is symmetric around the series center point x_0.

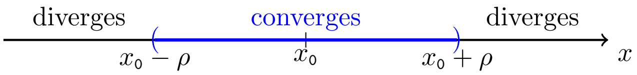


FIGURE 2. Example of the radius of convergence.

Example B.5. We state the radius of convergence of few power series. See [11, 13].

(1) The series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has radius of convergence $\rho = 1$.

(2) The series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $\rho = \infty$.

(3) The series $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,x^{(2n+1)}$ has radius of convergence $\rho = \infty$.

(4) The series $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,x^{(2n)}$ has radius of convergence $\rho = \infty$.

(5) The series $\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\,x^{(2n+1)}$ has radius of convergence $\rho = \infty$.

(6) The series $\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\,x^{(2n)}$ has radius of convergence $\rho = \infty$.

One of the most used tests for the convergence of a power series is the ratio test.

Theorem B.4 (Ratio Test). *Given the power series* $y(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^n$, *introduce the number* $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. *Then, the following statements hold:*

(1) The power series converges in the domain $|x-x_0|L < 1$.

(2) The power series diverges in the domain $|x-x_0|L > 1$.

(3) The power series may or may not converge at $|x-x_0|L = 1$.

Therefore, if $L \neq 0$, *then* $\rho = \dfrac{1}{L}$ *is the series radius of convergence; if* $L = 0$, *then the radius of convergence is* $\rho = \infty$.

Remark: The convergence of the power series at $x_0 + \rho$ and $x_0 - \rho$ needs to be studied on each particular case.

Power series are usually written using summation notation. We end this review mentioning a few summation index manipulations, which are fairly common. Take the series

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots,$$

which is usually written using the summation notation

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^n.$$

The label name, n, has nothing particular, any other label defines the same series. For example the labels k and m below,

$$y(x) = \sum_{k=0}^{\infty} a_k\,(x-x_0)^k = \sum_{m=-3}^{\infty} a_{m+3}\,(x-x_0)^{m+3}.$$

In the first sum we just changed the label name from n to k, that is, $k = n$. In the second sum above we relabel the sum, $n = m+3$. Since the initial value for n is $n = 0$, then the initial value of m is $m = -3$. Derivatives of power series can be computed derivating every term in the power series,

$$y'(x) = \sum_{n=0}^{\infty} n\,a_n\,(x-x_0)^{n-1} = \sum_{n=1}^{\infty} n\,a_n\,(x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + \cdots.$$

The power series for the y' can start either at $n = 0$ or $n = 1$, since the coefficients have a multiplicative factor n. We will usually relabel derivatives of power series as follows,

$$y'(x) = \sum_{n=1}^{\infty} n\,a_n\,(x-x_0)^{n-1} = \sum_{m=0}^{\infty} (m+1)\,a_{m+1}\,(x-x_0)^m$$

where $m = n-1$, that is, $n = m+1$.

C. Discrete and Continuum Equations

A differential equation is an equation, where the unknown is a function and at least one of its derivatives appears in the equation. Differential equations are essential for a mathematical description of nature—they lie at the core of many physical theories. In this section we show that differential equations can be obtained as a certain limit of difference equations.

We focus on a specific problem—a quantitative description of bacteria growth having unlimited space and food. We first measure the bacteria population at fixed time intervals, then we repeat the measurements at shorter and shorter time intervals. We write our measurements in a difference equation for a discrete time interval variable. We solve this difference equation, obtaining the bacteria population as a function of the initial population and the number of time intervals passed from the start of the experiment. We then compute a very particular limit on the difference equation, called the continuum limit. In this limit the time interval goes to zero and the number of time intervals goes to infinity so that their product remains constant. We will see that the continuum limit of the difference equation in this section is a differential equation, called the population growth differential equation.

C.1. The Difference Equation. We want to know how bacteria grows in time when they have unlimited space and food. To obtain such equation we observe—very carefully—how the bacteria grows. We put an initial amount of bacteria in a small region at the center of a petri dish, which is full of bacteria nutrients. In this way the bacteria population has unlimited space and food to grow for a certain time. The bacteria population is then proportional to the area in the petri dish covered in bacteria. With this setting we will perform several experiments in which we measure the bacteria population after regular time intervals.

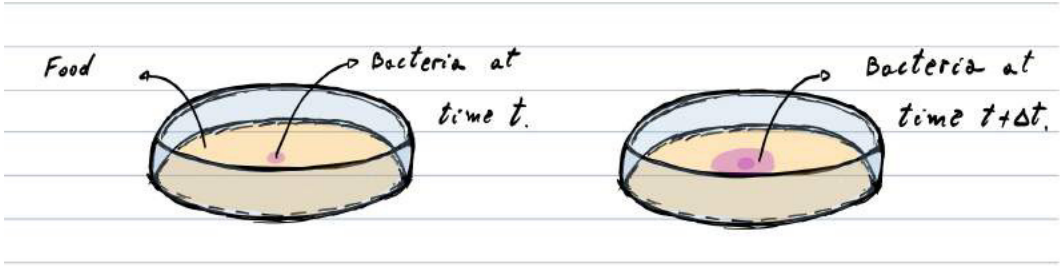

Food
Bacteria at time t.
Bacteria at time t+Δt.

Figure 3. Bacteria growth experiment with unlimited food and space.

First Experiment:

- fix the time interval between measurements by $\Delta t_1 = 1$ hour.
- denote the bacteria population after n time intervals as $P(n\Delta t_1) = P(n)$,
- introduce the initial bacteria population $P(0)$,

Our first measurement is $P(1)$, the bacteria population after 1 hour. It is convenient to write $P(1)$ as follows

$$P(1) = P(0) + \Delta P_1$$

where ΔP_1 is what we actually have measured, and it is the increment in bacteria population. In the same way we can write our first n measurements,

$$\begin{aligned} P(1) &= P(0) + \Delta P_1, \\ P(2) &= P(1) + \Delta P_2, \\ &\vdots \\ P(n) &= P((n-1)) + \Delta P_n, \end{aligned} \tag{C.1}$$

where ΔP_j, for $j = 1, \cdots, n$, is the increment in bacteria population at the measurement j relative to the measurement $j-1$. If you actually do the experiment—and if you look carefully enough at the ΔP_n carefully enough–you will find the following: *The increment in the bacteria population ΔP_n is not random, but it follows the rule*

$$\Delta P_n = K_1 \, P(n-1), \tag{C.2}$$

where K_1 depends on the type of bacteria and on the fact that we are measuring by $\Delta t_1 = 1$ hour. This last equation means that the growth of the bacteria population is proportional to the existing bacteria population. We use Eq. (C.2) in Eq. (C.1) and we get the formula

$$P(n) = P(n-1) + K_1 \, P(n-1), \qquad n = 1, 2, \cdots, N, \tag{C.3}$$

where N is the last time we measure, probably when the bacteria population fills the whole petri dish. This is the end of our first experiment.

Second Experiment: We reduce the time interval Δt when we take measurements. Now $\Delta t_2 = 30$ minutes, that is, $\Delta t_2 = 1/2$ hours. Since Δt_2 is no longer 1, we need to include it in the argument of P. If you carry out the experiment, you will find that Eq. (C.3) still holds for this case if we introduce $\Delta t_2 = \Delta t_1/2$ as follows,

$$P(n\Delta t_2) = P((n-1)\Delta t_2) + K_2 \, P((n-1)\Delta t_2), \qquad n = 1, 2, \cdots, N. \tag{C.4}$$

In this experiment we have to measure the new constant K_2. You will find that $K_2 = K_1/2$. This is reasonable, *the bacteria population grows in 30 minutes half it grows in one hour.* This is the end of our second experiment.

m-th Experiment: We now carry out many more similar experiments. For the m-th experiment we use a time interval $\Delta t_m = \Delta t_1/m$, where $\Delta t_1 = 1$ hour. If you carry out all these experiments, you will find the following relation,

$$P(n\Delta t_m) = P((n-1)\Delta t_m) + K_m \, P((n-1)\Delta t_m), \qquad n = 1, 2, \cdots, N, \tag{C.5}$$

where $K_m = K_1/m$.

By looking at all our experiments, we can see that *the constant K_m is in fact proportional to the time interval Δt_m used in the experiment, and the proportionality constant is the same for all experiments.* Indeed,

$$K_m = \frac{K_1}{m} \quad \Rightarrow \quad K_m = \frac{K_1}{\Delta t_1} \frac{\Delta t_1}{m} \quad \Rightarrow \quad K_m = r \, \Delta t_m,$$

where the constant $r = K_1/\Delta t_1$ depends only on the type of bacteria we are working with. Since the constant K_m in any of the experiments above is proportional to the time interval Δt_m used in each experiment, we can simplify the notation and discard the subindex m,

$$K = r \, \Delta t.$$

Then, the final conclusion of all our experiments is the following: the population of bacteria after n time intervals $\Delta t > 0$ is given by the equation

$$P(n\Delta t) = P((n-1)\Delta t) + r\,\Delta t\,P((n-1)\Delta t), \tag{C.6}$$

where r is a constant that depends on the type of bacteria studied and $n = 1, 2, \cdots$. This equation is a difference equation, because the argument of the population function takes discrete values. We call equation (C.6) the *discrete population growth equation*. The physical meaning of this constant r is given in the equation above,

$$r = \frac{\Delta P}{\Delta t}\,\frac{1}{P}$$

where $\Delta P = P(n\Delta t) - P((n-1)\Delta t)$ and $P = P((n-1)\Delta t)$. *So r is the rate of change in time of the bacteria population per bacteria*, that is, a relative rate of change.

C.2. Solving the Difference Equation. The difference equation (C.6) relates the bacteria population after n time intervals, $P(n\Delta t)$, with the bacteria population at the previous time interval, $P((n-1)\Delta t)$. To solve a difference equation means to find the bacteria population after n times intervals, $P(n\Delta t)$, in terms of the initial bacteria population, $P(0)$. The difference equation above can be solved, and the result is in the following statement.

Theorem C.1. *The difference equation*

$$P(n\Delta t) = P((n-1)\Delta t) + r\,\Delta t\,P((n-1)\Delta t),$$

relating $P(n\Delta t)$ with $P((n-1)\Delta t)$ has the solution

$$P(n\Delta t) = (1 + r\,\Delta t)^n\,P(0), \tag{C.7}$$

relating $P(n\Delta t)$ with $P(0)$.

Proof of Theorem C.1: Eq. (C.6) can be rewritten as

$$P(n\Delta t) = (1 + r\,\Delta t)\,P((n-1)\Delta t),$$

but we can also rewrite the expression for $P((n-1)\Delta t)$ in a similar way,

$$P((n-1)\Delta t) = (1 + r\,\Delta t)\,P((n-2)\Delta t),$$

and so on till we reach $P(0)$. Therefore,

$$\begin{aligned} P(n\Delta t) &= (1 + r\,\Delta t)\,P((n-1)\Delta t) \\ &= (1 + r\,\Delta t)^2\,P((n-2)\Delta t) \\ &\;\;\vdots \\ &= (1 + r\,\Delta t)^n\,P(0). \end{aligned}$$

So, we have solved the discrete equation for population growth, and the solution is

$$P(n\Delta t) = (1 + r\,\Delta t)^n\,P(0).$$

This establishes the Theorem. □

C.3. The Differential Equation. We want to know what happens to the difference equation (C.6) and its solutions (C.7) in the *continuum limit*:

$$\Delta t \to 0, \quad n\,\Delta t = t > 0 \quad \text{is constant.}$$

We call it the continuum limit because $\Delta t \to 0$, so we look more and more often at the bacteria population, and then $n \to \infty$, since we are making more and more observations. Rather than doing an experiment to find out what happens, we work directly with the discrete equation that models our bacteria population.

Theorem C.2. *The continuum limit of the discrete equation*

$$P(n\Delta t) = P((n-1)\Delta t) + r\,\Delta t\,P((n-1)\Delta t),$$

is the differential equation

$$P'(t) = r\,P(t). \tag{C.8}$$

Remark: The equation (C.8) is a differential equation because both P and P' appear in the equation. It is called the *exponential growth differential equation* because its solutions are exponentials that increase with time.

Proof of Theorem C.2: We start renaming n as $n+1$, then Eq. (C.6) has the form

$$P((n+1)\Delta t) = P(n\Delta t) + r\,\Delta t\,P(n\Delta).$$

From here it is simple to see that

$$P(n\Delta t + \Delta t) - P(n\Delta t) = r\,\Delta t\,P(n\Delta).$$

We now use that $n\,\Delta t = t$, then the equation above becomes

$$P(t + \Delta t) - P(t) = r\,\Delta t\,P(t).$$

Dividing by Δt we get

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t).$$

The continuum limit is given by $\Delta t \to 0$ and $n \to \infty$ such that $n\,\Delta t = t$ is constant. For each choice of t we have a particular limit. So we take such limit in the equation above,

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t).$$

Since t is held constant and $\Delta t \to 0$, the left-hand side above is the derivative of P with respect to t,

$$P'(t) = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t}.$$

So we get the differential equation

$$P'(t) = r\,P(t).$$

This establishes the Theorem. □

C.4. Solving the Differential Equation. We now find all solutions to the exponential growth differential equation in (C.8). By a solution we mean a function P that depends on time such that its derivative is r times the function itself.

Theorem C.3. *All the solutions of the differential equation $P'(t) = r\,P(t)$ are*

$$P(t) = P_0\, e^{rt}, \tag{C.9}$$

where P_0 is a constant.

Remark: The constant P_0 in (C.9) is the initial population, $P(0) = P_0$.

Proof of Theorem C.3: To find all solutions we start dividing the equation by P,

$$\frac{P'(t)}{P(t)} = r.$$

We now integrate both sides with respect to time,

$$\int \frac{P'(t)}{P(t)}\, dt = \int r\, dt.$$

The integral on the right-hand side is simple to do, we need to integrate a constant,

$$\int \frac{P'(t)}{P(t)}\, dt = rt + c_0,$$

where c_0 is an arbitrary constant. On the left-hand side we can introduce a substitution

$$p = P(t) \quad \Rightarrow \quad dp = P'(t)\, dt.$$

Then, the the equation above becomes

$$\int \frac{dp}{p} = rt + c_0.$$

The integral above is simple to do and the result is

$$\ln|p| = rt + c_0.$$

We now replace back $p = P(t)$, and we can solve for P,

$$\ln|P(t)| = rt + c_0 \quad \Rightarrow \quad |P(t)| = e^{rt+c_0} = e^{kt}\, e^{c_0} \quad \Rightarrow \quad P(t) = (\pm e^{c_0})\, e^{rt}.$$

We denote $c = (\pm e^{c_0})$, then all the solutions to the exponential growth equation,

$$P(t) = c\, e^{rt}, \qquad c \in \mathbb{R}.$$

The constant c is the initial population. Indeed, given an initial population P_0, called an *initial condition*, then it fixes the constant c, because

$$P_0 = P(0) = c\, e^0 = c \quad \Rightarrow \quad c = P_0.$$

Then the solution of the differential equation with an initial population P_0 is

$$P(t) = P_0\, e^{rt}.$$

This establishes the Theorem. □

Remark: We see that the solution of the differential equation is an exponential, which is the origin of the name for the differential equation.

C.5. Summary and Consistency. By carefully observing how bacteria grow when they have unlimited space and food we came up with a diference equation, Eq. (C.6). We were able to solve this difference equation and the result was Eq. (C.7). We then studied what happened with the difference equation in the continuum limit—we look at the bacteria at infinitely short time intervals. The result is a differential equation, the exponential growth differential equation (C.8). Recalling calculus ideas we were able to find all solutions of this differential equation, given in Eq. (C.9). We can summarize all this as follows,

$$\begin{tikzcd}
\textbf{Discrete description} & \Delta t \to 0 & \textbf{Continuous description} \\
P(n\Delta t) = (1 + r\,\Delta t)\,P((n-1)\Delta t) \arrow[rr] \arrow[d, "\text{Soving the equation}"] & & P'(t) = r\,P(t) \arrow[d, "\text{Solving the equation}"] \\
P(n\Delta t) = (1 + r\,\Delta t)^n\,P_0 \arrow[rr, "\text{Consistency}"] & & P(t) = P_0\,e^{rt}
\end{tikzcd}$$

We are now going to show the consistency of the solutions. We have a solution of the discrete equation, we have a solution of the continuum equation, and now we show that the continuum limit of the former is the latter.

Theorem C.4 (Consistency). *The continuum limit of the solutions of the difference equations are the solutions of the differential equation,*

$$P(n\Delta t) = (1 + r\,\Delta t)^n\,P_0 \quad \longrightarrow \quad P(t) = P_0\,e^{rt}.$$

Proof of Theorem C.4: We start with the discrete solution given in Eq. (C.7),

$$P(n\Delta t) = (1 + r\Delta t)^n\,P_0, \tag{C.10}$$

and we recall that $t = n\Delta t$, hence $\Delta t = t/n$. So we write

$$P(t) = \Big(1 + \frac{rt}{n}\Big)^n\,P_0.$$

Now we need to study the limit of the expression above as $n \to \infty$ while t is constant in that limit. This is a good time to remember the Euler number e,

$$e = \lim_{n\to\infty}\Big(1 + \frac{1}{n}\Big)^n$$

satisfies that

$$e^x = \lim_{n\to\infty}\Big(1 + \frac{x}{n}\Big)^n.$$

Using the formula above for $x = rt$ we get

$$\lim_{n\to\infty}\Big(1 + \frac{rt}{n}\Big)^n = e^{rt}.$$

With all this we can write the continuum limit as

$$P(t) = \lim_{n\to\infty}\Big(1 + \frac{rt}{n}\Big)^n\,P_0 = e^{rt}\,P_0.$$

But the function

$$P(t) = P_0\,e^{rt}$$

is the solution of the differential equation obtained using methods from calculus. This establishes the Theorem. □

In the following examples we provide a table with data from different physical systems. Then we find the difference equation that describes such data and its solution. After that we compute the continuum limit, which gives us the differential equation for that system. We finally solve the differential equation.

Example C.1. The population of bees in a state, given in thousands, is given by

Year	2000	2002	2004	2006	2008	2010
Population	2	10	50	250	1250	6250

Model these data using exponential growth model, denoting by $P(t)$ the bee population in thousands at time t, with time in years since the year 2000. For example, for the year 2008, the variable t is 8. Consider a discrete model for the data in the table above given by

$$P((n+1)\Delta t) = P(n\Delta t) + k\,\Delta t\,P(n\Delta t).$$

(1) Determine the growth-rate coefficient k using the data for the years 2000 and 2002.
(2) Determine the growth-rate coefficient k again, this time using the data for the years 2008 and 2010.
(3) Use the value of k found above to write the discrete equation describing the bee population. Write Δt instead of the time interval in the table.
(4) Solve the discrete equation for the bee population.
(5) Find the continuum differential equation satisfied by the bee population and write the initial condition for this equation.
(6) Find all solutions of the continuum equation found in part (5).

Solution:

(1) The growth coefficient computed using the years 2000 and 2002 is

$$k = \frac{(10-2)}{(2002-2000)}\,\frac{1}{2} \quad\Rightarrow\quad k = 2.$$

(2) The growth coefficient computed using the years 2008 and 2010 is

$$k = \frac{(6250-1250)}{(2010-2008)}\,\frac{1}{1250} \quad\Rightarrow\quad k = 2.$$

(3) We now use $k = 2$ and Δt arbitrary to write the discrete equation that describes the data in the table. We denote

$$P(n+1) = P((n+1)\Delta t), \qquad Pn = P(n\Delta t),$$

then, the discrete equation is

$$P(n+1) = Pn + r\,\Delta t\,Pn,$$

which is the analogous to Eq. (C.6).

(4) Since

$$\left.\begin{aligned} Pn &= (1+r\,\Delta t)\,P(n-1),\\ P(n-1) &= (1+r\,\Delta t)\,P(n-2),\end{aligned}\right\} \quad\Rightarrow\quad Pn = (1+r\,\Delta t)^2\,P(n-2),$$

repeating this argument till we reach P_0 we get

$$Pn = (1+r\,\Delta t)^n\,P_0.$$

(5) The continuum equation is obtained from the discrete equation taking the continuum limit:

$$\Delta t \to 0, \quad n \to \infty \quad \text{such that} \quad n\,\Delta t = t \in \mathbb{R}.$$

Using the discrete equation in (3) we get

$$P(n+1) - Pn = r\,\Delta t\,Pn \quad \Rightarrow \quad \frac{P(n+1) - Pn}{\Delta t} = r\,Pn.$$

If we write what $P(n+1)$ and Pn actually are, we get

$$\frac{P(n\Delta t + \Delta t) - P(n\,\Delta t)}{\Delta t} = r\,P(n\,\Delta t).$$

Since $n\,\Delta t = t$, we replace it above,

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = r\,P(t).$$

Since $\Delta t \to 0$ we get the continuum equation

$$P'(t) = r\,P(t).$$

(6) To solve the continuum equation we rewrite it as follows,

$$\frac{P'}{P} = r \quad \Rightarrow \quad \int \frac{P'(t)}{P(t)}\,dt = \int r\,dt \quad \Rightarrow \quad \ln(|P|) = rt + c_0,$$

where $c_0 \in \mathbb{R}$ is an arbitrary integration constant, and we $\ln(|P|)' = P'/P$. Then,

$$P(t) = \pm e^{rt+c_0} = \pm e^{c_0}\,e^{rt}, \quad \text{denote} \quad c_1 = \pm e^{c_0} \quad \Rightarrow \quad P(t) = c_1\,e^{rt}, \quad c_1 \in \mathbb{R}.$$

The constant c_1 is determined by the initial population $P(0) = P_0$. Indeed

$$P_0 = P(0) = c_1\,e^0 = c_1 \quad \Rightarrow \quad c_1 = P_0$$

therefore we get that

$$P(t) = P_0\,e^{rt}.$$

◁

Example C.2. A bacteria population increases by a factor $(1 + 8\,\Delta t)$ in a time period Δt. Every Δt we harvest an amount of bacteria $20\,\Delta t$.

(a) Write the **discrete equation** that relates the bacteria population at $(n+1)\,\Delta t$ with the bacteria population at $n\,\Delta t$.

(b) Find the **continuum limit** in the discrete equation found in part (a) above. Recall that the continuum limit is $n \to \infty$ and $\Delta t \to 0$ so that $n\,Dt = t$ is constant in that limit.

(c) Solve the differential equation in part (b) in the case there is an **initial population** of 100 bacteria.

Solution:

(a) We know that the bacteria population P increases by a factor $(1 + 8\,\Delta t$ during the time interval Δt. **If we forget that we harvest bacteria**, then after $(n+1)$ time intervals the bacteria population is

$$P((n+1)\Delta t) = (1 + 8\,\Delta t)\,P(n\,\Delta t).$$

The equation above does not include the fact that we harvest $20\,\Delta t$ bacteria every time interval Δt. If we include this fact we get the equation

$$P((n+1)\Delta t) = (1 + 8\,\Delta t)\,P(n\,\Delta t) - 20\,\Delta t,$$

where the negative sign in the last term indicates we reduce the population by that amount when we harvest.

(b) The continuum limit is computed as folllows: If we see a product $n\,\Delta t$, we replace it by t, that is,

$$P(n\,\Delta t + \Delta t) = (1 + 8\,\Delta t)\,P(n\,\Delta t) - 20\,\Delta t \quad \Leftrightarrow \quad P(t + \Delta t) = (1 + 8\,\Delta t)\,P(t) - 20\,\Delta t.$$

We now reorder terms such that we get an incremental quotient on the left-hand side,

$$P(t + \Delta t) - P(t) = 8\,\Delta t\,P(t) - 20\,\Delta t \quad \Rightarrow \quad \frac{P(t + \Delta t) - P(t)}{\Delta t} = 8\,P(t) - 20.$$

Now we take the limit $\Delta t \to 0$ keeping t constant, and we get the continuum equation

$$P'(t) = 8\,P(t) - 20.$$

(c) We now need to solve the differential equation above. We do the same calculation we did in the case of zero harvesting.

$$P'(t) = 8\,P(t) - 20 \quad \Rightarrow \quad \frac{P'(t)}{(8\,P(t) - 20)} = 1 \quad \Rightarrow \quad \int \frac{P'(t)}{(8\,P(t) - 20)}\,dt = \int dt.$$

On the left-hand side above we substitute $u = 8\,P(t) - 20$, so $du = 8\,p'(t)\,dt$. Then,

$$\int \frac{1}{u}\,\frac{du}{8} = \int dt \quad \Rightarrow \quad \frac{1}{8}\ln|u| = t + c_1 \quad \Rightarrow \quad \ln|8\,P(t) - 20| = 8t + 8c_1.$$

We compute the exponential of both sides,

$$|8\,P(t) - 20| = e^{8t + 8c_1} = e^{8t}\,e^{8c_1} \quad \Rightarrow \quad 8\,P(t) - 20 = (\pm e^{8c_1})\,e^{8t}.$$

If we call $c_2 = (\pm e^{8c_1})$, we get that

$$8\,P(t) - 20 = c_2\,e^{8t} \quad \Rightarrow \quad P(t) = \frac{c_2}{8}\,e^{8t} + \frac{20}{8},$$

and again relabeling the constant $c = c_2/8$ we get that

$$P(t) = c\,e^{8t} + \frac{5}{2}.$$

We know that at time $t = 0$ we have $P(0) = 100$ bacteria, which fixes the constant c, because

$$100 = P(0) = c\,e^0 + \frac{5}{2} = c + \frac{5}{2} \quad \Rightarrow \quad c = 100 - \frac{5}{2} = \frac{195}{2}.$$

So the continuum formula for the bacteria population is

$$P(t) = \frac{195}{2}\,e^{8t} + \frac{5}{2}.$$

◁

C.6. Exercises.

9.3.1.- The fish population in a lake, given in hundred thousands, is given by

Year	2000	2001	2002	2003	2004	2005
Population	3	4.5	6.75	10.125	15.1875	22.78125

Model these data using exponential growth model, denoting by $P(t)$ the fish population in hundred thousands at time t, with time in years since the year 2000. For example, for the year 2005, the variable t is 5. Consider a discrete model for the data in the table above given by

$$P((n+1)\Delta t) = P(n\Delta t) + k\,\Delta t\,P(n\Delta t).$$

(1) Determine the growth-rate coefficient k using the data for the years 2000 and 2001.
(2) Determine the growth-rate coefficient k again, this time using the data for the years 2004 and 2005.
(3) Use the value of k found above to write the discrete equation describing the fish population. Write Δt instead of the time interval in the table.
(4) Solve the discrete equation for the fish population.
(5) Find the continuum differential equation satisfied by the fish population and write the initial condition for this equation.
(6) Solve the the initial value problem found in the previous part.
(7) Use a computer to compare the solutions to the discrete equation (with any $\Delta t \neq 0$) and continuum equation for the fish population.

9.3.2.- A bacteria population increases by a factor r in a time period Δt. Every Δt we harvest an amount of bacteria $P_h\,\Delta t$, where P_h is a fixed constant.

(a) Write the discrete equation that relates the bacteria population at $(n+1)\,\Delta t$ with the bacteria population at $n\,\Delta t$. This equation is similar, but not equal, to Eq. (C.6) above.
(b) Find the continuum limit in the discrete equation found in part (a) above. Recall that the continuum limit is $n \to \infty$ and $\Delta t \to 0$ so that $n\,\Delta t = t$ is constant in that limit. Denote by $P(t)$ the bacteria population at the time t.
(c) Solve the differential equation in part (b) in the case there is an initial population of P_0 bacteria.
(d) The solution of the continuum differential equation in part (b) above also holds in the case that the initial population of bacteria is **smaller** than P_h/r. So, consider the case where $P_0 < P_h/r$ and find the time t_1 such that the bacteria population vanishes.

9.3.3.- The amount of a radioactive material **decreases** by a factor $r = 1/2$ in a time period Δt.

(a) Write the discrete equation that relates the amount of radioactive material at $(n+1)\,\Delta t$ with the radioactive material at $n\,\Delta t$. This equation is similar, but not equal, to Eq. (C.6) above.
(b) What is the main difference between a radioactive decay system and a bacteria population system?
(c) Take the continuum limit in the discrete equation found in part (a) above. Recall that the continuum limit is $n \to \infty$ and $\Delta t \to 0$ so that $n\,\Delta t = t$ is constant in that limit. Denote by $N(t)$ the amount of radioactive material at the time t. You should obtain the radioactive decay differential equation.

(d) Solve the radioactive decay differential equation. Denote by N_0 the initial amount of the radioactive material.

(e) The half-life of a radioactive material is the time τ such that $N(\tau) = \dfrac{N(0)}{2}$. Find the half-life of radiative material in this problem. Find an equation relating the half life τ with the radioactive decay constant r.

D. Review Exercises

Coming up.

E. Practice Exams

Coming up.

F. Answers to exercises

Chapter 1: First Order Equations

Section 1.1: Linear Constant Coefficient Equations

1.1.1.- $y' = 5y + 2$.

1.1.2.- $a = 2$ and $b = 3$.

1.1.3.- $y = c\,e^{3t}$, for $c \in \mathbb{R}$.

1.1.4.- $y(t) = c\,e^{-4t} + \dfrac{1}{2}$, with $c \in R$.

1.1.5.- $y(t) = c\,e^{2t} - \dfrac{5}{2}$.

1.1.6.- $y(x) = \dfrac{9}{2}\,e^{-4t} + \dfrac{1}{2}$, with $c \in R$.

1.1.7.- $y(x) = \dfrac{5}{3}\,e^{3t} + \dfrac{2}{3}$.

1.1.8.- $\psi(t, y) = \Big(y + \dfrac{1}{6}\Big)\,e^{-6t}$

$$y(x) = c\,e^{-6t} - \frac{1}{6}.$$

1.1.9.- $y(t) = \dfrac{7}{6}\,e^{-6t} - \dfrac{1}{6}$.

1.1.10.- * Not given.

Section 1.2: Linear Variable Coefficient Equations

1.2.1.- $y(t) = c\,e^{2t^2}$.

1.2.2.- $y(t) = c\,e^{-t} - e^{-2t}$, with $c \in \mathbb{R}$.

1.2.3.- $y(t) = 2e^{t} + 2(t-1)\,e^{2t}$.

1.2.4.- $y(t) = \dfrac{\pi}{2t^2} - \dfrac{\cos(t)}{t^2}$.

1.2.5.- $y(t) = c\,e^{t^2(t^2+2)}$, with $c \in \mathbb{R}$.

1.2.6.- $y(t) = \dfrac{t^2}{n+2} + \dfrac{c}{t^n}$, with $c \in \mathbb{R}$.

1.2.7.- $y(t) = 3\,e^{t^2}$.

1.2.8.- $y(t) = c\,e^{t} + \sin(t) + \cos(t)$, for all $c \in \mathbb{R}$.

1.2.9.- $y(t) = -t^2 + t^2\sin(4t)$.

1.2.10.- Define $v(t) = 1/y(t)$. The equation for v is $v' = tv - t$. Its solution is $v(t) = c\,e^{t^2/2} + 1$. Therefore,

$$y(t) = \frac{1}{c\,e^{t^2/2} + 1}, \quad c \in \mathbb{R}.$$

1.2.11.- $y(x) = \big(6 + c\,e^{-x^2/4}\big)^2$

1.2.12.- $y(x) = \big(4\,e^{3t} - 3\big)^{1/3}$

1.2.13.- * Not given.

Section 1.3: Separable Equations

1.3.1.- Implicit form: $\dfrac{y^2}{2} = \dfrac{t^3}{3} + c$.

Explicit form: $y = \pm\sqrt{\dfrac{2t^3}{3} + 2c}$.

1.3.2.- $y^4 + y + t^3 - t = c$, with $c \in \mathbb{R}$.

1.3.3.- $y(t) = \dfrac{3}{3 - t^3}$.

1.3.4.- $y(t) = c\,e^{-\sqrt{1+t^2}}$.

1.3.5.- $y(t) = t\,\big(\ln(|t|) + c\big)$.

1.3.6.- $y^2(t) = 2t^2\,\big(\ln(|t|) + c\big)$.

1.3.7.- Implicit: $y^2 + ty - 2t = 0$.
Explicit: $y(t) = \dfrac{1}{2}\big(-t + \sqrt{t^2 + 8t}\big)$.

1.3.8.- Hint: Recall the Definition 1.3.4 and use that

$$y_1'(x) = f\big(x, y_1(x)\big),$$

for any independent variable x, for example for $x = kt$.

1.3.9.- * Not Given.

Section 1.4: Exact Equations

1.4.1.-

(a) The equation is exact. $N = (1+t^2)$, $M = 2t\,y$, so $\partial_t N = 2t = \partial_y M$.

(b) Since a potential function is given by $\psi(t,y) = t^2\,y + y$, the solution is

$$y(t) = \frac{c}{t^2+1}, \qquad c \in \mathbb{R}.$$

1.4.2.-

(a) The equation is exact. We have $N = t\cos(y) - 2y$, $M = t + \sin(y)$,

$$\partial_t N = \cos(y) = \partial_y M.$$

(b) Since a potential function is given by $\psi(t,y) = \dfrac{t^2}{2} + t\sin(y) - y^2$, the solution is

$$\frac{t^2}{2} + t\sin(y(t)) - y^2(t) = c,$$

for $c \in \mathbb{R}$.

1.4.3.-

(a) The equation is exact. We have $N = -2y + t\,e^{ty}$, $M = 2 + y\,e^{ty}$,

$$\partial_t N = (1 + t\,y)\,e^{ty} = \partial_y M.$$

(b) Since a potential function is given by $\psi(t,y) = 2t + e^{ty} - y^2$, the solution is

$$2t + e^{t\,y(t)} - y^2(t) = c,$$

for $c \in \mathbb{R}$.

1.4.4.-

(a) $\mu(x) = 1/x$.

(b) $y^3 - 3xy + \dfrac{18}{5}\,x^5 = 1$.

1.4.5.-

(a) $\mu(x) = x^2$.

(b) $y^2(x^4 + 1/2) = 2$.

(c) $y(x) = -\dfrac{2}{\sqrt{1+2x^4}}$. The negative square root is selected because the the initial condition is $y(0) < 0$.

1.4.6.-

(a) The equation for y is not exact. There is no integrating factor depending only on x.

(b) The equation for $x = y^{-1}$ is not exact. But there is an integrating factor depending only on y, given by

$$\mu(y) = e^y.$$

(c) An implicit expression for both $y(x)$ and $x(y)$ is given by

$$-3x\,e^{-y} + \sin(5x)\,e^y = c,$$

for $c \in \mathbb{R}$.

1.4.7.- * Not Given.

Section 1.5: Applications

1.5.1.-

(a) Denote $m(t)$ the material mass as function of time. Use m in mgr and t in hours. Then

$$m(t) = m_0\, e^{-kt},$$

where $m_0 = 50$ mgr and $k = \ln(5)$ hours.

(b) $m(4) = \dfrac{2}{25}$ mgr.

(c) $\tau = \dfrac{\ln(2)}{\ln(5)}$ hours, so $\tau \simeq 0.43$ hours.

1.5.2.-

(a) We know that $(\Delta T)' = -k\,(\Delta T)$, where $\Delta T = T - T_s$ and the cooler temperature is $T_s = 3$ C, while k is the liquid cooling constant. Since $T_s' = 0$,

$$T' = -k\,(T - 3).$$

(b) The integrating factor method implies $(T' + k\,T)e^{kt} = 3k\,e^{kt}$, so

$$\big(T\,e^{kt}\big)' - \big(3\,e^{kt}\big)' = 0.$$

Integrating we get $(T-3)\,e^{kt} = c$, so the general solution is $T = c\,e^{-kt} + 3$. The initial condition implies $18 = T(0) = c + 3$, so $c = 15$, and the function temperature is

$$T(t) = 15\,e^{-kt} + 3.$$

(c) To find k we use that $T(3) = 13$ C. This implies $13 = 15\,e^{-3k} + 3$, so we arrive at

$$e^{-3k} = \frac{13-3}{15} = \frac{2}{3},$$

which leads us to $-3k = \ln(2/3)$, so we get

$$k = \frac{1}{3}\,ln(3/2).$$

1.5.3.- Since

$$Q(t) = Q_0\,e^{-(r_o/V_0)t},$$

the condition

$$Q_1 = Q_0\,e^{-(r_o/V_0)t_1}$$

implies that

$$t_1 = \frac{V_0}{r_o}\,\ln\Big(\frac{Q_0}{Q_1}\Big).$$

Therefore, $t_1 = 20\,\ln(5)$ minutes.

1.5.4.- Since

$$Q(t) = V_0 q_i\,\big(1 - e^{-(r_o/V_0)t}\big)$$

and

$$\lim_{t\to\infty} Q(t) = V_0 q_i,$$

the result in this problem is

$$Q(t) = 300\,\big(1 - e^{-t/50}\big)$$

and

$$\lim_{t\to\infty} Q(t) = 300 \text{ grams}.$$

1.5.5.- Denoting $\Delta r = r_i - r_o$ and $V(t) = \Delta r\,t + V_0$, we obtain

$$Q(t) = \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}} Q_0 + q_i\,\Big[V(t) - V_0\Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}}\Big].$$

A reordering of terms gives

$$Q(t) = q_i V(t) - \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}} (q_i V_0 - Q_0)$$

and replacing the problem values yields

$$Q(t) = t + 200 - 100\,\frac{(200)^2}{(t+200)^2}.$$

The concentration $q(t) = Q(t)/V(t)$ is

$$q(t) = q_i - \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}+1}\Big(q_i - \frac{Q_0}{V_0}\Big).$$

The concentration at $V(t) = V_m$ is

$$q_m = q_i - \Big[\frac{V_0}{V_m}\Big]^{\frac{r_o}{\Delta r}+1}\Big(q_i - \frac{Q_0}{V_0}\Big),$$

which gives the value

$$q_m = \frac{121}{125} \text{ grams/liter}.$$

In the case of an unlimited capacity, $\lim_{t\to\infty} V(t) = \infty$, thus the equation for $q(t)$ above says

$$\lim_{t\to\infty} q(t) = q_i.$$

Section 1.6: Nonlinear Equations

1.6.1.-

$$\begin{aligned} y_0 &= 0, \\ y_1 &= t, \\ y_2 &= t + 3t^2, \\ y_3 &= t + 3t^2 + 6t^3. \end{aligned}$$

1.6.2.-

(a) $$\begin{aligned} y_0 &= 1, \\ y_1 &= 1 + 8t, \\ y_2 &= 1 + 8t + 12\,t^2, \\ y_3 &= 1 + 8t + 12\,t^2 + 12\,t^3. \end{aligned}$$

(b) $c_k(t) = \frac{8}{3}\,3^k t^k$.

(c) $y(t) = \frac{8}{3}e^{3t} - \frac{5}{3}$.

1.6.3.-

(a) Since $y = \sqrt{y_0^2 - 4t^2}$, and the initial condition is at $t = 0$, the solution domain is

$$D = \Big[-\frac{y_0}{2}, \frac{y_0}{2}\Big].$$

(b) Since $y = \dfrac{y_0}{1 - t^2 y_0}$ and the initial condition is at $t = 0$, the solution domain is

$$D = \Big[-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}}\Big].$$

1.6.4.-

(a) Write the equation as

$$y' = -\frac{2\ln(t)}{(t^2 - 4)}\,y.$$

The equation is not defined for

$$t = 0 \quad t = \pm 2.$$

This provides the intervals

$$(-\infty, -2),\ (-2, 2),\ (2, \infty).$$

Since the initial condition is at $t = 1$, the interval where the solution is defined is

$$D = (0, 2).$$

(b) The equation is not defined for

$$t = 0, \quad t = 3.$$

This provides the intervals

$$(-\infty, 0),\ (0, 3),\ (3, \infty).$$

Since the initial condition is at $t = -1$, the interval where the solution is defined is

$$D = (-\infty, 0).$$

1.6.5.-

(a) $y = \frac{2}{3}t$.

(b) Outside the disk $t^2 + y^2 \leqslant 1$.

Chapter 2: Second order linear equations

Section 2.1: Variable Coefficients

2.1.1.- .

2.1.2.- .

Section 2.2: Reduction Order Methods

2.2.1.-

(a) $v' = -(3/t)v + 3/t^2$.
(b) $v(t) = 3/(2t)$.
(c) $y(t) = (3/2\ln(t) + 3$.

2.2.2.-

(a) $\dot{w} = -3w/y$.
(b) $w(1) = 5$.
(c) $w(y) = 5/y^3$.
(d) $y'(t) = 5/(y(t))^3$, with $y(01) = 1$.
(e) $y(t) = (20t + 1)^{1/4}$.

2.2.3.- $y(t) = 7t/6 + 1)^{1/7}$.

2.2.4.- $y_2(t) = c/t^4$, with $c \in \mathbb{R}$.

2.2.5.- * Not given.

Section 2.3: Homogeneous Constant Coefficient Equations

2.3.1.-

(a) $r_+ = 4$, $r_- = 3$.
(b) $y_+(t) = e^{4t}$, $y_-(t) = e^{3t}$.
(c) $y(t) = -4\,e^{4t} + 5\,e^{3t}$.

2.3.2.-

5

(a) $r_+ = 4 + 3i$, $r_- = 4 - 3i$.
(b) $y_+(t) = e^{4t}\cos(3t)$,
$y_-(t) = e^{4t}\sin(3t)$.
(c) $y(t) = 2\,e^{4t}(\cos(3t) - \sin(3t))$.

2.3.3.-

(a) $r_+ = r_- = 3$.
(b) $y_+(t) = e^{3t}$, $y_-(t) = t\,e^{3t}$.
(c) $y(t) = e^{3t}(1 + t)$.

2.3.4.- * Not given.

Section ??: Repeated Roots

??.??.- .

??.??.- .

Section ??: Undetermined Coefficients

??.??.- .

??.??.- .

Section ??: Variation of Parameters

??.??.- .

??.??.- .

Chapter 3: Power Series Solutions

Section 3.1: Regular Points

3.1.1.- .

3.1.2.- .

Section 2.4: The Euler Equation

2.4.1.- .

2.4.2.- .

Section 3.2: Regular-Singular Points

3.2.1.- .

3.2.2.- .

Chapter 4: The Laplace Transform

Section 4.1: Introduction to the Laplace Transform

4.1.1.-

(a) $I_N = -\Big(\frac{1}{N} - \frac{1}{4}\Big).$

(b) $I = \frac{1}{4}.$

4.1.2.-

(a) $I_N = -\frac{1}{s}\big(e^{-sN} - e^{-5s}\big).$

(b) $I = \dfrac{e^{-5s}}{s}$, for $s > 0$. I diverges for $s \leqslant 0$.

4.1.3.-

(a) $I_N = -\frac{1}{(s-2)}\big(e^{-(s-2)N} - 1\big).$

(b) $F(s) = \frac{1}{(s-2)}$, for $s > 2$.

4.1.4.-

(a)

$$I_N = -\frac{N\,e^{-(s+2)N}}{(s+2)} - \frac{1}{(s+2)^2}\big(e^{-(s+2)N} - 1\big).$$

(b) $F(s) = \dfrac{1}{(s+2)^2}$, for $s > 2$.

4.1.5.-

(a)

$$I_N = -\frac{s^2\,e^{-sN}}{s^2+2^2}\Big(\frac{1}{s}\sin(2N) + \frac{2}{s^2}\cos(2N)\Big) + \frac{2}{s^2+2^2}.$$

(b) $F(s) = \dfrac{2}{s^2+4}$, for $s > 0$.

4.1.6.-

(a)

$$I_N = \frac{s^2\,e^{-sN}}{s^2+4^2}\Big(-\frac{1}{s}\cos(4N) + \frac{4}{s^2}\sin(4N)\Big) + \frac{s}{s^2+4^2}.$$

(b) $F(s) = \dfrac{s}{s^2+16}$, for $s > 0$.

4.1.7.-

$\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2-a^2}$, for $s > |a|$.

4.1.8.- *

$\mathcal{L}[\cosh(at)] = \dfrac{s}{s^2-a^2}$, for $s > |a|$.

Section 4.3: Discontinuous Sources

4.3.??.- .

4.3.??.- .

Section 4.4: Generalized Sources

4.4.1.- * Not Given.

4.4.2.- .

Section 4.5: Convolution Solutions

??.??.- .

??.??.- .

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Chapter 5: Systems of Linear Differential Equations

Section ??: Introduction

??.??.- . **??.??.-** .

Section 8.1: Systems of Algebraic Equations

8.1.1.- . **8.1.2.-** .

Section 8.2: Matrix Algebra

8.2.1.- . **8.2.2.-** .

Section 5.1: Linear System of Differential Equations

5.1.??.- . **5.1.??.-** .

Section 8.3: Diagonalizable Matrices

8.3.1.- . **8.3.2.-** .

Section ??: Constant Coefficients Systems

??.??.- . **??.??.-** .

Chapter 7: Boundary Value Problems

Section 7.1: Eigenvalue-Eigenfunction Problems

7.1.1.- .

7.1.2.- .

Section 7.2: Overview of Fourier Series

7.2.1.- .

7.2.2.- .

Section 7.3: Applications: The Heat Equation

7.3.1.-

7.3.2.- .

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